Applications of BCFW recursion relation

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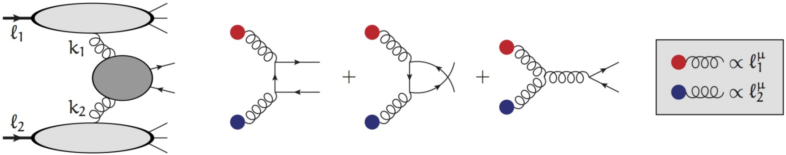
w Large Hadron Collider - teoria i fenomenologia"

Motivation

High-energy factorisation (Catani,Ciafaloni,Hautmann, 1991 / Collins,Ellis, 1991)



$$\sigma_{h_1,h_2\to q\bar{q}} = \int d^2k_{1\perp}\frac{dx_1}{x_1}\, f_g(x_1,k_{1\perp})d^2k_{2\perp}\frac{dx_2}{x_2}\, f_g(x_2,k_{2\perp})\,\hat{\sigma}_{gg}\left(\frac{m^2}{x_1x_2s},\frac{k_{1\perp}}{m},\frac{k_{2\perp}}{m}\right)$$

where the f_g's are the unintegrated gluon densities, obeying BFKL, BK, CCFM or KGBJS evolution equations.

The partonic process

$$g^*g^* \to q\bar{q}$$

is described by **a gauge invariant amplitude** in the high-energy kinematics

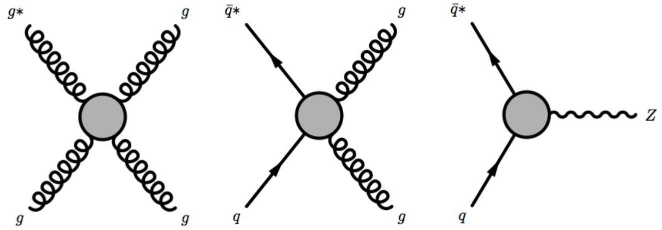
$$k_1^\mu = x_1\, l_1^\mu + k_{1\perp}^\mu \quad , \quad k_2^\mu = x_2\, l_2^\mu + k_{2\perp}^\mu \quad \text{for} \quad l_{1,2}\cdot k_{1,2\perp} = 0$$

despite the fact that the two initial state gluons are off shell, due to the transverse component of their momentum.

Is it possible to have gauge-invariant expressions for other scattering processes with off shell partons, maybe to devise a general method to compute them ?

The question is physically motivated by possible phenomenological applications such as

HE factorisation $\begin{cases} \text{production of forward dijets initiated with gluons} & : \quad gg^* \to gg \\ \text{production of forward dijets initiated with quarks} & : \quad q\bar{q}^* \to gg \\ \text{production of a } Z \text{ boson by a quark-antiquark pair} & : \quad q\bar{q}^* \to Z \end{cases}$



Computing scattering amplitudes in Yang-Mills theories via ordinary Feynman diagrams: soon overwhelming !

Number of Feynman diagrams at tree level:

# of gluons	4	5	6	7	8	9	10
# of diagrams	4	25	220	2485	34300	559405	10525900

A method to compute helicity amplitudes which avoids the evaluation of a huge number of Feynman diagrams

BCFW recursion relation(s) (ever since 2005)

Britto, Cachazo, Feng, Nucl.Phys. B715 (2005) 499-522
Britto, Cachazo, Feng, Witten, Phys.Rev.Lett. 94 (2005) 181602

The spinor helicity formalism

High energy limit ⇒ massless particles ⇒ Weyl basis for spinors.

If $p^2 = 0$, it can be cast in the Pauli matrices language,

$$p \cong p^\mu \sigma_\mu = \begin{pmatrix} p^0 - p^3 & -p^1 + i\,p^2 \\ -p^1 - i\,p^2 & p^0 + p^3 \end{pmatrix} = |p]\langle p|$$

$$|p] = \begin{pmatrix} L(p) \\ \mathbf{0} \end{pmatrix} \qquad L(p) = \frac{1}{\sqrt{|p^0 + p^3|}} \begin{pmatrix} -p^1 + i\,p^2 \\ p^0 + p^3 \end{pmatrix}$$

$$|p\rangle = \begin{pmatrix} \mathbf{0} \\ R(p) \end{pmatrix} \qquad R(p) = \frac{\sqrt{|p^0 + p^3|}}{p^0 + p^3} \begin{pmatrix} p^0 + p^3 \\ p^1 + i\,p^2 \end{pmatrix}$$

and the "dual" spinors

$$[p| = \left((\mathcal{E} L(p))^T \,,\, \mathbf{0} \right) \qquad \langle p| = \left(\mathbf{0}\ (\mathcal{E}^T R(p))^T \right) \qquad \text{where} \quad \mathcal{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Important: unified formalism for bosons and spinors

BCFW recursion relation

Two very simple ideas:

- if the amplitude is formally treated as a function of a complex variable z and it vanishes for $z \to \infty$, then the integral extended to an infinite contour enclosing all poles vanishes

$$\lim_{z\to\infty} \mathcal{A}(z) = 0 \Rightarrow \frac{1}{2\pi i} \oint dz\, \frac{\mathcal{A}(z)}{z} = 0$$

  implying that the value at $z = 0$ (physical amplitude) can be determined as a sum of the residues at the poles:

$$\mathcal{A}(0) = -\sum_i \frac{\lim_{z\to z_i}[(z - z_i)\, f(z)]}{z_i}$$

  where z_i is the location of the i-th pole
- Poles in Yang-Mills tree level amplitudes can only be due to gluon propagators dividing the n-point amplitude into two on-shell sub-amplitudes with $k+1$ and $n-k+1$ gluons

To properly "complexify" $\mathcal{A}$: for helicities $(h_1, h_n) = (-,+)$ (no loss of generality...)

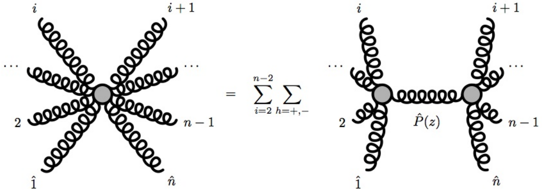
$$
\begin{aligned}
|1] &\rightarrow |\hat{1}] \equiv |1] - z\,|n] \Rightarrow p_1 \rightarrow \hat{p}_n = |1]\langle 1| - z|1]\langle n| \\
|n\rangle &\rightarrow |\hat{n}\rangle \equiv |n\rangle + z|1\rangle \Rightarrow p_n \rightarrow \hat{p}_n = |n]\langle n| + z|1]\langle n|
\end{aligned}
$$

On-shellness (and thus gauge invariance) and momentum conservation preserved throughout.

The result is the recursive relation

$$
\mathcal{A}(g_1,\ldots,g_n) = \sum_{i=2}^{n-2} \sum_{h=+,-} \mathcal{A}(g_1,\ldots,g_i,\hat{P}^h)\,\frac{1}{(p_1+\cdots+p_i)^2}\,\mathcal{A}(-\hat{P}^{-h},g_{i+1},\ldots,g_n)
$$

$$
z_i = \frac{(p_1+\cdots+p_i)^2}{[1|p_1+\cdots+p_i|n\rangle}
$$
location of the pole corresponding for the "i-th" partition



The inclusion of fermions

M. Luo, C. Wen, JHEP 0503 (2005) 004
M. Luo, C. Wen, Phys.Rev. D71 (2005) 091501

Despite the fact that BCFW was originally proved for pure Yang-Mills theories, it can be easily extended to fermions:

- factorisation properties of amplitudes when intermediate particles go on-shell is a general consequence of unitarity
- the other problem is the behaviour for $z \to \infty$, but either a twistor space result (Cachazo,Svrcek and Witten JHEP 0409 (2004) 006) or a smart choice of reference lines helps to overcome the problem, so that $\lim_{z\to\infty} \mathcal{A}(z) = 0$ holds

Formulas for MHV amplitudes with n gluons and n gluons plus a $q\bar{q}$ pair:

$$\mathcal{A}(g_1^+, g_2^+, \ldots, g_i^-, \ldots, g_j^-, \ldots, g_n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \ldots \langle n-1\ n\rangle\langle n1\rangle}$$

$$\mathcal{A}(q^-, g_1^-, g_2^+, \ldots, g_n^+, \bar{q}^+) = \frac{\langle q1\rangle^3\langle \bar{q}1\rangle}{\langle \bar{q}q\rangle\langle q1\rangle\langle 12\rangle \ldots \langle n\bar{q}\rangle}$$

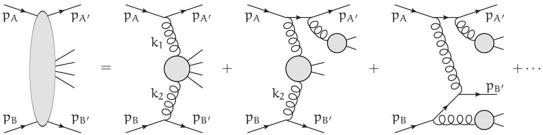
The problem is the following: how to calculate gauge-invariant amplitudes with off-shell partons in a gauge invariant way in the high-energy kinematics ?

One idea...

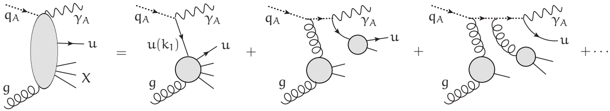
on-shell amplitudes are gauge invariant, so off-shell gauge-invariant amplitudes could be got by embedding them into on-shell processes...

...two results:

- for off-shell gluons: represent g^* as coming from a $\bar{q}qg$ vertex, with the quarks taken to be on-shell



- for off-shell quarks: represent q^* as coming from a $\gamma\bar{q}q$ vertex, with a 0 momentum and $\bar{q}$ on shell (and vice-versa)

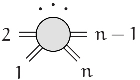


K. Kutak, P. Kotko, A. van Hameren, JHEP 1301 (2013) 078
K. Kutak, T. Salwa, A. van Hameren, Phys.Lett. B727 (2013) 226-233

Despite Feynman rules are devised, the diagrammatic approach is too slow to allow for amplitudes containing more than 4 particles.
It is natural to ask whether something like a BCFW recursion relation exists with off-shell particles...

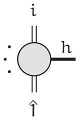
A generalisation of the BCFW approach to include off-shell gluons was devised in 2014
A. van Hameren, JHEP 1407 (2014) 138

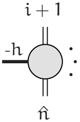
All the possible poles for pure Yang-Mills amplitudes with off-shell gluons
($k_i \equiv$ i-th momentum, $p_i^2 = 0$ $p_i \cdot k_i = 0$, $q_i \equiv$ auxiliary momentum):

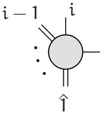


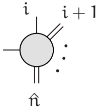
$$= \sum_{i=2}^{n-2} \sum_{h=+,-} \mathrm{A}_{i,h} + \sum_{i=2}^{n-1} \mathrm{B}_i + \mathrm{C} + \mathrm{D} \,,$$

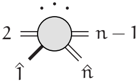
where

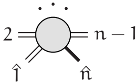
$\mathrm{A}_{i,h} =$ 

$\dfrac{1}{(\sum_{j=1}^{i} k_j)^2}$ 

$\mathrm{B}_i =$ 

$\dfrac{1}{2\, p_i \cdot \sum_{j=1}^{i} k_j}$ 

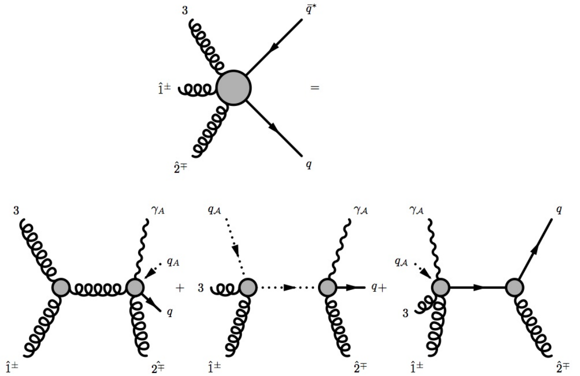
$\mathrm{C} = \dfrac{\langle q_1 p_1 \rangle}{\langle q_1 | \not{k}_1 | p_1]}$ 

$\mathrm{D} = \dfrac{[p_n q_n]}{\langle p_n | \not{k}_n | q_n]}$ 

Preliminary results

The 5-point functions we aim at computing need just 3-point functions to be reproduced recursively.

An interesting phenomenon shows up here: if we pick up two gluons as reference lines, we find that that the eikonal Feynman rules for off-shell fermion imply **the need for a diagram with one off-shell gluon** (second term on the r.h.s.)



Below are the 3-point functions with 1 off-shell parton needed for the recursive computation of these two 5-point scattering amplitudes

$$q^*\, g \to q\, g\, g$$
$$q\, \bar{q}^* \to g\, g\, g$$

$$\mathcal{A}(\bar{q}^*, q^-, g_1^+) = \frac{\langle q\, \bar{q}\rangle}{\langle q|\not{k}_{\bar{q}}|\bar{q}]} \frac{[\bar{q}1]^3\, [q1]}{[\bar{q}q]\, [q1]\, [1\bar{q}]}$$

$$\mathcal{A}(\bar{q}^*, q^+, g_1^-) = \frac{[q\bar{q}]}{\langle \bar{q}|\not{k}_{\bar{q}}|q]} \frac{\langle \bar{q}1\rangle^3\, \langle q1\rangle}{\langle \bar{q}q\rangle\, \langle q1\rangle\, \langle 1\bar{q}\rangle}$$

$$\mathcal{A}(g_1^*, g_2^+, g_3^-) = \frac{[12]}{\langle p_1|\not{k}_1|2]} \frac{\langle 31\rangle^3}{\langle 12\rangle\langle 23\rangle}$$

$$\mathcal{A}(g_1^*, g_2^-, g_3^+) = \frac{[12]}{\langle p_1|\not{k}_1|2]} \frac{\langle 21\rangle^3}{\langle 13\rangle\langle 32\rangle}$$

where $p_1^2 = 0\,,\ k_1 \cdot p_1 = 0.$

Summary and conclusions

- the BCFW recursion relation was originally invented to allow for the efficient computation of on-shell scattering amplitudes with many (≥ 5) particles in Yang-Mills theories. It works magnificently.
- High-energy factorisation requires gauge invariant scattering amplitudes with off-shell partons, both gluons and quarks.
- Embedding tricks to provide such amplitudes have been devised recently: they work really well, but suffer from the same kind of computational slow-down when it comes to many particle amplitudes
- the BCFW construction was thus extended to the case of pure Yang Mills, accounting for off-shell gluons: this implies identifying new kinds of poles in the amplitudes
- the same identification of the supplementary poles (w.r.t. the on-shell case) has been worked out for amplitudes with gluons and off-shell fermions and some 5 point amplitudes are on their way to us... **with Andreas van Hameren**

THANK YOU FOR YOUR ATTENTION