REMARKS ON MATHEMATICAL FOUNDATIONS OF QUANTUM MECHANICS

Paweł Klimasara
(in collaboration with Jerzy Król)

University of Silesia

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THE PLAN OF THE PRESENTATION

1. QM vs classical theories
2. Foundations of mathematics
3. Model theory and QM
4. Algebraic structure of QM
5. Topos theory and physics
6. "Classical" QM?
7. Discussion and perspectives

HOW TO MAKE QM "MORE CLASSICAL"?

One quite obvious difference between QM and classical theories is the lack of simultaneous measurement of non-commuting observables.

THE KOCHEN - SPECKER THEOREM (1967)

In QM based on Hilbert spaces of states of dimension > 2 there is no global assignement (before and after the measurement) of real values to observables.

QUANTUM MECHANICS

No global valuation for non-commuting observables.

CLASSICAL THEORIES

Observables have preassigned values.

Ambiguity of contextual theories

To find out what is the reflection of such ambiguity on a formal level we need to take a closer look at foundations of mathematics itself.

FUNDAMENTAL AMBIGUITY IN MATHEMATICS

THE ZFC SET THEORY:

Foundation of (almost) all of mathematics.

A MODEL OF A THEORY

A mathematical universe in which the theorems of this theory are fulfilled.

THE ZFC SET THEORY HAS INFINITELY MANY NON-ISOMORPHIC MODELS - THERE IS NO UNIQUE UNIVERSE FOR MATHEMATICS!

Kurt Gödel (1906 - 1978)

What is the essential difference between non-isomorphic models of ZFC?

FORMAL TOOLS OF MATHEMATICAL LOGIC

It appears that different models of ZFC has different sets of reals, i.e. some formal object can be a real number in one model, while in another model it is not.

IN 1964 P. COHEN SOLVED FAMOUS CONTINUUM HYPOTHESIS PROBLEM USING NEW AND VERY POWERFUL TECHNIQUE - THE FORCING METHOD

The crucial fact here is that forcing is a formal tool to extend given model of ZFC, roughly speaking by adding some of the "missing" reals.

Paul Cohen (1934 - 2007)

But do we need abstract methods of model theory in QM?

PROPOSAL OF USING FORCING IN QM

Paul A. Benioff (1976)

THERE IS NO SINGLE MODEL OF ZFC THAT CAN BE USED FOR THE CORRECT REPRESENTATION OF MATHEMATICS OF QM AND ITS STATISTICAL PREDICTIONS

So if we want to use standard foundations of mathematics to describe QM, we rather need to focus on some "dynamics" between many models of ZFC.

Paul A. Benioff

Forcing - formal representation of a measurement?

We have agreement with the Kochen-Specker theorem if we change mathematical universe after the measurement. Missing real values in a starting model assigned to the observables during a measurement are present in a new model after the measurement.

ALGEBRAIC STRUCTURES IN PHYSICS

John von Neumann (1932)

QM can be described via non-commutative C^*-algebras of observables.

On the other hand classical theories can be represented by commutative C^*-algebras of observables.

John von Neumann (1903 - 1957)

If one could find a mathematical universe (model) where a non-commutative C^*-algebra becomes commutative, then QM would be described by a classical theory.

DOES SUCH MATHEMATICAL UNIVERSE EXIST?

ANSWER IN NEW FOUNDATIONS OF MATHEMATICS?

"Standard" mathematics - based on set theory (ZFC).

Basic structure: a set. [static element]

"New" mathematics - based on category theory.

Basic structure: a category, i.e. objects linked by arrows. [dynamical sets]

Some special categories, called toposes, play crucial role in our considerations.

Banaschewski - Mulvey theorem (2006)

For every non-commutative C^*-algebra of operators one can construct a topos within which this structure becomes **commutative** C^*-algebra.

Common results of two approaches.

Interesting by itself is the fact that toposes are the categories based on forcing logic - they describe the variability of models of ZFC.

COSTS OF MAKING QM "CLASSICAL" (IN THE TOPOS)

HOWEVER SUBSTITUTING SET THEORY WITH CATEGORY THEORY FORCES SWITCHING CLASSICAL LOGIC INTO INTUITIONISTIC LOGIC WHICH CAUSES SOME SERIOUS FORMAL CHANGES

Consequences of using toposes:

- No proofs by a contradiction,
- The lack of the law of excluded middle,
- No axiom of choice.

THE LAW OF EXCLUDED MIDDLE

$$(p \vee \neg p) \leftrightarrow 1$$

IN PRINCIPLE SUCH "CLASSICAL" QM WOULD BE A TOOL FOR MAKING QUANTUM-MECHANICAL CALCULATIONS AS IF THEY WERE CLASSICAL.

DISCUSSION AND PERSPECTIVES

TOPOSES IN PHYSICS (based on set theory models connected by forcing relations):

- Alternative for set-theoretical approach,
- Schools of Landsman and Isham.

THE TOVARIANCE RULE (N.P. Landsman, 2007)

Physical theories should be independent of the choice of a topos - there is always a possibility of (formal) "translation" of a "classical" QM based on some topos to a QM based on models of usual set theory [ZFC] (and back).

POSSIBILITIES (open problems so far)
In the spectral topos:

- **CLASSICAL CALCULATIONS IN QM?**
- **CLASSICAL MEASUREMENTS IN QM?**

Interesting relation between QM and a modern history of mathematics.

Thank You for a Patient Hearing.