A No-Hair Theorem for Non-Abelian Extreme Horizons

Carmen Li

The University of Edinburgh

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Main Result

A No-Hair Thoerem

Any stationary near horizon solution to the extremal Einstein-Yang-Mills black hole with a non-positive cosmological constant in 4-D is that of Kerr-Newman and therefore unique.

Motivation

1. As mentioned by Romanowska and Kaneda the LHC is looking for microscopic black holes
2. Extremal black holes ($\kappa = 0$) do not radiate via Hawking Radiation $T_H = \kappa/2\pi$
3. Increasing evidence for instability of extremal black holes (Lucietti, Reall '12)
4. All SUSY black holes are extremal
5. All known extremal black holes have AdS_2 symmetry in their near-horizon geometries
6. AdS_2 structure fundamental to quantum theory of extremal back holes from AdS/CFT correspondence
7. Kerr/CFT correspondence (Guica et al. '08)

Motivation

1. Black hole uniqueness does not apply to EYM black holes

   (Smoller, Wesserman and Yau '93)

2. 4-D $SU(2)$ EYM theory with $\Lambda < 0$ is a consistent truncation of 11-D SUGRA on S^7 (Pope '85)

3. Existence of AdS_2 near-horizon symmetry proved for any extremal black hole with rotational symmetry in Einstein-Maxwell theory (Kunduri, Lucietti, Reall '07)

4. We extend this to the *non-abelian* Einstein-Yang-Mills (EYM) theory with $\Lambda \leq 0$

5. 4-D $SU(2)$ EYM theory with $\Lambda < 0$ is a consistent truncation of 11-D SUGRA on S^7 (Pope '85)

Near-Horizon Coordinates

1. In the N-H limit the extremal black hole metric in Gaussian null coordinates is

$$ds^2 = r^2 F(x) dv^2 + 2dvdr + 2rh_a(x)dvdx^a + \gamma_{ab}(x)dx^a dx^b$$

   - $r = 0$ is the Killing horizon $\mathcal{N}$ of the null vector $V = \frac{\partial}{\partial v}$
   - x^a are coordinates on the closed 2-D spatial cross-section $\mathcal{H}$

2. N-H metric is invariant under the 2-D non-abelian isometry group G_2

   - translation $v \to v + constant$ generated by ∂_v
   - "dilation" $r \to \epsilon r$ and $v \to v/\epsilon$ generated by $v\partial_v - r\partial_r$

Near-Horizon Equations

1. Einstein Equation

$$R_{\mu\nu} = 2 \ \mathrm{Tr}\left(\mathcal{F}_{\mu}{}^{\delta}\mathcal{F}_{\nu\delta} - \frac{1}{4}g_{\mu\nu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\right) + \Lambda g_{\mu\nu}$$

2. Yang-Mills 2-form $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ satisfies the Y-M field equation $D \star \mathcal{F} = 0$
3. In N-H limit $\mathcal{F}$ takes the form

$$\mathcal{F} = \mathcal{F}_{vr}(x)dv \wedge dr + r\mathcal{F}_{va}(x)dv \wedge dx^a + \hat{\mathcal{F}}$$

4. The most general gauge field in N-H limit is

$$\mathcal{A} = r\Delta(x)dv + \hat{\mathcal{A}}$$

5. Hats denote quantities on $\mathcal{H}$
6. For any $\mathfrak{g}$-valued form $\mathcal{D}X = dX + [\mathcal{A}, X]$

Near-Horizon Equations (cont.)

1. The EYM system reduces to these equations defined purely on $\mathcal{H}$

$$
\begin{aligned}
\hat{R}_{ab} &= \frac{1}{2}h_a h_b - \hat{\nabla}_{(a}h_{b)} + \Lambda\gamma_{ab} \\
&\quad + \mathrm{Tr}\left(2\hat{\mathcal{F}}_{ac}\hat{\mathcal{F}}_b{}^c + \Delta^2\gamma_{ab} - \frac{1}{2}\hat{\mathcal{F}}_{cd}\hat{\mathcal{F}}^{cd}\gamma_{ab}\right) \\
F &= \frac{1}{2}h_a h^a - \frac{1}{2}\hat{\nabla}_a h^a + \Lambda - \mathrm{Tr}\left(\Delta^2 + \frac{1}{2}\hat{\mathcal{F}}_{ab}\hat{\mathcal{F}}^{ab}\right) \\
\hat{D}\star_2\hat{\mathcal{F}} &= \star_2(\hat{D}\Delta - h\Delta) + \star_2 i_h\hat{\mathcal{F}} \\
\hat{\nabla}_a F &= F h_a + 2h_b\hat{\nabla}_{[a}h_{b]} - \hat{\nabla}_b\hat{\nabla}_{[a}h_{b]} \\
&\quad - 2\mathrm{Tr}[(\hat{\mathcal{F}}_{ab} + \Delta\gamma_{ab})(\hat{D}^b\Delta - h^b\Delta)]
\end{aligned}
$$

Stationary Horizon

1. All stationary rotating black holes are axisymmetric according to rigidity theorem
2. Stationary N-H metric therefore admits $G_2 \times U(1)$ isometry
3. Thus $\mathcal{H} \cong S^2$ or T^2
4. The metric on $\mathcal{H}$ can be written as

$$\gamma_{ab} dx^a dx^b = \frac{dx^2}{B(x)} + B(x) d\phi^2$$

5. The metric data h can be written as

$$h = -\frac{\Gamma'(x)}{\Gamma(x)} dx + \frac{B(x) k^\phi(x)}{\Gamma(x)} d\phi$$

for some functions $k^\phi(x)$ and $\Gamma(x) > 0$

General Equations (cont.)

1. We choose the gauge $\mathcal{L}_\phi \hat{\mathcal{F}} = 0$ such that $\hat{\mathcal{F}} = a'(x) dx \wedge d\phi$ with $\hat{\mathcal{A}} = a(x) d\phi$
2. $\hat{R}_{x\phi}$ Einstein equation implies $k^\phi = k = constant$
3. Writing
$$A = \Gamma^2 F - k^{\phi 2} B \ , \qquad R = \frac{r}{\Gamma}$$
the 4-D N-H metric takes the form
$$ds^2 = R^2 A dv^2 + 2\Gamma dv dR + \frac{1}{B} dx^2 + B(d\phi + kR dv)^2$$
4. $k = 0$ corresponds to static case
5. Remaining $\hat{R}_{ab}$ components gives Γ is a quadratic function of x therefore T^2 topology is ruled out

Key Identities

1. The Bianchi identity is equivalent to

$$B\left(\frac{A}{\Gamma}\right)' = 4\Gamma\ \mathrm{Tr}(\Delta[a, a'])$$

2. In Einstein-Maxwell theory the RHS vanishes thus A is proportional to Γ; it can be shown $A = -C^2\Gamma$ therefore the isometry group for the metric is enhanced to $SO(2,1) \times U(1)$:

$$ds^2 = \Gamma(-C^2R^2dv^2 + 2dvdR) + \frac{1}{B}dx^2 + B(d\phi + kRdv)^2$$

3. Define the *obstruction term* $T \equiv \Gamma^2\mathrm{Tr}\left(\Delta[a, a']\right)$ and $e^2 \equiv \Gamma^2\mathrm{Tr}(a'^2 + \Delta^2)$, Y-M field equations imply

$$\begin{aligned} BT' &= -\Gamma^2\mathrm{Tr}\left([a, a']^2 + [a, \Delta]^2\right) \\ B(e^2)' &= -4T \end{aligned}$$

Main Result

1. $B\ \partial/\partial x$ is a globally defined vector field which vanishes at the poles (x_1, x_2) of S^2
2. $B(e^2)' = -4T$ implies $T(x_1) = T(x_2) = 0$
3. Also have $BT' \leq 0$ with $BT'(x_1) = BT'(x_2) = 0$
4. Suppose there is one point in (x_1, x_2) where $BT' < 0$
5. Then $T' < 0$ and the fundamental theorem of calculus states that

$$T(x_2) - T(x_1) = \int_{x_1}^{x^2} T' dx < 0$$

   which contradicts with $T(x_1) = T(x_2) = 0$.
6. Hence $T = 0$ for all $x_1 \leq x \leq x_2$ and the obstruction term vanishes
7. Find $A_0 = A/\Gamma < 0$ from integrating the R_{+-} Einstein equation for $\Lambda \leq 0$

No-hair Theorem

Summary

We have shown that the near horizon geometry for a stationary and axisymmetric extremal black hole in the Einstein-Yang-Mills theory with $\Lambda \leq 0$ is given by

$$ds^2 = \Gamma(-C^2 r^2 dv^2 + 2dvdr) + \frac{1}{B(x)}dx^2 + B(x)(d\phi + krdv)^2$$

which admits an enhanced $SO(2,1) \times U(1)$ isometry, where the metric function $B(x)$ of the spatial cross-section $\mathcal{H} \cong S^2$ is

$$B\Gamma = \frac{\beta g^2}{4}x^4 + \left(\frac{6g^2k^2}{\beta} - C^2\right)x^2 + \frac{4}{\beta^3}(C^2\beta k^2 - 3g^2k^4 - e^2\beta^2) \ .$$

This is isometric to the near-horizon limit of the extremal Kerr-Newman(-AdS_4) metric.

Thank you

Staticity Conditions

1. The timelike Killing vector is hypersurface orthogonal on a static Killing horizon
$$V \wedge dV = 0$$
2. This is equivalent to
$$dh = 0 \qquad dF = Fh \qquad \hat{D}\Delta - \Delta h = 0$$
3. Define θ such that $h = d\log\theta^{-2}$, Y-M field equation gives
$$\hat{D}(\theta^{-2} \star_2 \hat{\mathcal{F}}) = \hat{D}(\theta^{-2}\Delta) = 0$$
hence $b \equiv \theta^{-2} \star_2 \hat{\mathcal{F}}$ and $e \equiv \theta^{-2}\Delta$ are covariantly constant
4. Einstein equation implies
$$2\hat{\nabla}^2\theta = -\frac{4}{3}\Lambda\theta + 4\theta^{-3}\mathrm{Tr}\left(b^2 + e^2\right) + C\theta^{-2}$$
θ has two types of solutions: constant or non-constant θ

Static Solutions

1. For non-constant θ one can check that $\mathcal{H}$ cannot have S^2 topology by requiring the absence of conical singularities
2. For constant θ N-H metric becomes a direct product of $AdS_2 \times \mathcal{H}$

$$ds^2 = -\left(\mathrm{Tr}(\Delta^2 + (\star_2 \hat{\mathcal{F}})^2) + |\Lambda|\right) r^2 dv^2 + 2dvdr + \gamma_{ab} dx^a dx^b$$

3. Einstein equation implies $\mathcal{H}$ is one of the maximally symmetric spaces

$$\hat{R}_{ab} = \left(\mathrm{Tr}(\Delta^2 + (\star_2 \hat{\mathcal{F}})^2) + \Lambda\right) \gamma_{ab}$$

4. $AdS_2 \times S^2$ is N-H limit of 4-D extremal Reissner-Nordström-AdS_4
5. $\Lambda < 0$ can be asymptotically *locally* AdS where topological censorship allows $\mathcal{H}$ to have higher genus

Backup Slide

1. Reduction of 11-D SUGRA

$$ds_{11}^2 = ds_4^2 + 2g^{-2}d\xi^2 + \frac{1}{2}g^{-2}\sum_i \left[\cos^2\xi(\sigma^i - gA^i)^2 + \sin^2\xi(\tilde{\sigma}^i - gA^i)^2\right]$$