General formulation of transverse hydrodynamics

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Outline

- Introduction
- Hydrodynamical equations for transversally thermalized matter
- Thermodynamics of two-dimensional systems
- Lorentz structure of the phase-space distribution function and its moments
- Conclusions

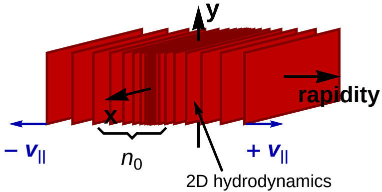
Based on: R. Ryblewski, W. Florkowski, Phys. Rev. **C77**, 064906 (2008)

Transversally thermalized matter

- Motivation ...
  - Evolution of the system created in heavy-ion collisions at RHIC energies best described by hydrodynamics of an almost perfect fluid
  - Standard hydrodynamics assumes three-dimensional (3D) thermalization
  - **Puzzle**: this approach demands early thermalization to describe v_2, BUT such fast local equilibration is hard to achieve with elastic perturbative cross-sections
  - **Possible solution**: at early stages of the evolution, the hydrodynamic approach applies only to the transverse degrees of freedom
- ... and origin of the model
  - Concept initially formulated by Heinz and Wong, PRC 66 (2002) 014907
  - Different implementation of this idea given recently by Bialas et al., Phys. Lett., B661 (2008) 325

Transversally thermalized matter

- System is a superposition of non-interacting transverse clusters
- Cluster is formed by particles moving with the same value of rapidity
- Single cluster is described by 2D hydrodynamics
- n_0 describes density of clusters in rapidity



- We use the following definitions of rapidity and spacetime rapidity ...

$$y = \frac{1}{2}\ln\frac{E+p_{||}}{E-p_{||}} \qquad \eta = \frac{1}{2}\ln\frac{t+z}{t-z}$$

- ... and standard parameterization of the four-momentum and spacetime coordinate

$$\mathrm{p}^{\mu} = \left(E, \vec{p}_{\perp}, p_{||}\right) = \left(m_{\perp}\cosh y, \vec{p}_{\perp}, m_{\perp}\sinh y\right)$$
$$\mathrm{x}^{\mu} = \left(t, \vec{x}_{\perp}, z\right) = \left(\tau\cosh\eta, \vec{x}_{\perp}, \tau\sinh\eta\right)$$
$$\tau = \sqrt{t^2 - z^2} \qquad m_{\perp} = \sqrt{m^2 + p_x^2 + p_y^2}$$

Energy-momentum conservation laws

- Hydrodynamic equations for transversally thermalized matter follow from energy-momentum conservation laws

$$\partial_\mu T^{\mu\nu} = 0$$

- Formalism is based on specific form of the $T^{\mu\nu}$ tensor

$$T^{\mu\nu} = \frac{n_0}{\tau}\left[(\varepsilon_2 + P_2)\, U^\mu U^\nu - P_2\,(g^{\mu\nu} + V^\mu V^\nu)\right]$$

$$\mathrm{T}^{\mu\nu}_{LRF} = \tfrac{n_0}{\tau}\mathrm{diag}(\varepsilon_2, P_2, P_2, 0)$$

- The four vectors U^μ and V^μ are defined by the equations

$$\begin{aligned} U^\mu &= (u_0 \cosh\eta, u_x, u_y, u_0 \sinh\eta) \\ V^\mu &= (\sinh\eta, 0, 0, \cosh\eta) \end{aligned}$$

where u_0,u_x,u_y are the components of the four-velocity of the fluid element in the rest frame of the cluster

$$u^\mu = (u_0,\, u_x,\, u_y, 0) = \left(u_0,\, \vec{u}_\perp, 0\right)$$

$$u_0^2 - \vec{u}_\perp^2 = 1 \quad u_0 = \left(1 - v^2\right)^{-\frac{1}{2}}$$

Adiabaticity of the flow

- Energy-momentum conservation laws lead to the entropy conservation

$$U_\nu \partial_\mu T^{\mu\nu} = T \partial_\mu S^\mu = 0$$

where the entropy current is defined by the expression

$$S^\mu = \frac{n_0}{\tau} s_2 U^\mu \qquad \varepsilon_2 + P_2 = T s_2$$

- The use of the entropy conservation in energy-momentum conservation laws leads to the analog of the Euler equation

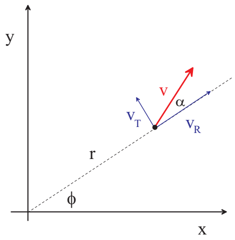
$$U^\mu \partial_\mu \left(T U^\nu \right) = \partial^\nu T + V^\nu V^\mu \partial_\mu T$$

- The dynamics of the system is specified by three independent equations for four unknown functions s_2, T, u_x, u_y. The **EOS**, which specifies $s_2(T)$, is needed to close the system.

Cylindrical coordinates

- We introduce cylinrical coordinates r and ϕ

$$r = \sqrt{r_x^2 + r_y^2} \qquad \mathbf{x}_\perp = (r_x, r_y)$$
$$\phi = \arctan\left(r_y/r_x\right)$$



- Parameterization of the fluid velocity takes form

$$v_x = v\cos(\alpha+\phi) \qquad u_x = u_0 v_x$$
$$v_y = v\sin(\alpha+\phi) \qquad u_y = u_0 v_y$$

- Hydrodynamic equations may be written explicitly in the form

$$\frac{\partial}{\partial\tau}\left(r s_2 u_0\right) + \frac{\partial}{\partial r}\left(r s_2 u_0 v\cos\alpha\right) + \frac{\partial}{\partial\phi}\left(s_2 u_0 v\sin\alpha\right) = 0$$
$$\frac{\partial}{\partial\tau}\left(rTu_0 v\right) + r\cos\alpha\frac{\partial}{\partial r}\left(Tu_0\right) + \sin\alpha\frac{\partial}{\partial\phi}\left(Tu_0\right) = 0$$
$$Tu_0^2 v\left(\frac{d\alpha}{d\tau} + \frac{v\sin\alpha}{r}\right) - \sin\alpha\frac{\partial T}{\partial r} + \frac{\cos\alpha}{r}\frac{\partial T}{\partial\phi} = 0$$

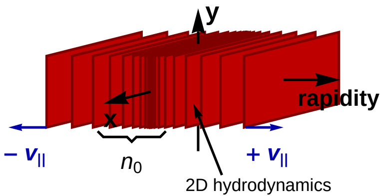
the derivative $d/d\tau$ denotes the total derivative with respect to time

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau} + v\cos\alpha\frac{\partial}{\partial r} + \frac{v\sin\alpha}{r}\frac{\partial}{\partial\phi}$$

Breaking of boost-invariance

The conservation of the tensor $T^{\mu\nu}$ implies the conservation of the tensor $T^{\mu\nu}_{(\eta)}$

$$\partial_\mu T^{\mu\nu} = 0 \Rightarrow \partial_\mu T^{\mu\nu}_{(\eta)} = \partial_\mu \left[n_0(\eta) T^{\mu\nu} \right] = 0$$



The density of the transverse clusters n_0 may depend on spacetime rapidity η. It means that formalism is not necessarily boost-invariant.

Landau matching conditions

- Expected transition 2D → 3D hydrodynamics may be described by assuming the Landau matching conditions at the transition point

$$T^{\mu\nu} U_\nu = T^{\mu\nu}_{3D} U_\nu$$

where $T^{\mu\nu}_{3D}$ is the standard energy-momentum tensor of relativistic hydrodynamics of perfect fluid

$$T^{\mu\nu}_{3D} = (\varepsilon + P) U^\mu U^\nu - P g^{\mu\nu}$$

The equivalent condition is local conservation of energy and momentum at the transition point

$$\frac{n_0}{\tau} \varepsilon_2 U^\mu = \varepsilon U^\mu$$

- Additional condition is the entropy production

$$\frac{n_0}{\tau} s_2 < s$$

Such transition described in terms of dissipative hydrodynamics by P.Bożek

2D thermodynamic quantities

- 2D thermodynamic densities follow directly from 2D potential Ω of non-interacting bosons (upper signs) or fermions (lower signs)

$$\Omega(T, V_2, \mu) = \pm \nu_g T V_2 \int \frac{d^2 p_\perp}{(2\pi)^2} \ln\left(1 \mp e^{(\mu - m_\perp)/T}\right)$$

$$\text{gluon dominated systems} \rightarrow \nu_g = 8\,(\textit{gluon states}) \times 2\,(\textit{spin} \uparrow\downarrow)$$

- number of particles not conserved $\rightarrow \quad \mu = 0$

$$N_2 = \nu_g V_2 \int \frac{d^2 p_\perp}{(2\pi)^2}\, g \qquad P_2 = \nu_g \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{p_\perp^2}{2 m_\perp}\, g$$

$$S_2 = -\nu_g V_2 \int \frac{d^2 p_\perp}{(2\pi)^2} \left[g \ln g \mp (1 \pm g) \ln(1 \pm g)\right]$$

$$E_2 = \nu_g V_2 \int \frac{d^2 p_\perp}{(2\pi)^2}\, m_\perp\, g$$

- the equilibrium distribution function

$$g(m_\perp) = \frac{1}{e^{m_\perp/T} \mp 1}$$

A. Massless fermions and bosons

(*fermions*)

$$n_2 = \frac{\nu_g \pi T^2}{24}, \qquad \varepsilon_2 = \frac{3\nu_g \zeta(3) T^3}{4\pi}$$

(*bosons*)

$$n_2 = \frac{\nu_g \pi T^2}{12}, \qquad \varepsilon_2 = \frac{\nu_g \zeta(3) T^3}{\pi}$$

$$P_2 = \frac{1}{2}\varepsilon_2, \quad c_s^2 = \frac{\partial P_2}{\partial \varepsilon_2} = \frac{1}{2}.$$

B. Classical limit (finite masses)

$$\begin{aligned} n_2 &= \frac{\nu_g T}{2\pi}(m+T)e^{-m/T} \\ P_2 &= \frac{\nu_g T^2}{2\pi}(m+T)e^{-m/T} \\ s_2 &= \frac{\nu_g}{2\pi}[m^2+3mT+3T^2]e^{-m/T} \\ \varepsilon_2 &= \frac{\nu_g T}{2\pi}[T^2+(m+T)^2]e^{-m/T} \\ c_s^2 &= \frac{T(m^2+3mT+3T^2)}{m^3+3m^2T+6mT^2+6T^3} \end{aligned} \quad \underset{m\to 0}{\lim} \longrightarrow \quad \begin{aligned} n_2 &= \frac{\nu_g T^2}{2\pi} \\ P_2 &= \frac{\nu_g T^3}{2\pi} \\ s_2 &= \frac{3\nu_g T^2}{2\pi} \\ \varepsilon_2 &= \frac{\nu_g T^3}{\pi} \\ c_s^2 &= \frac{1}{2} \end{aligned}$$

Lorentz structure of the distribution function

- Factorization of the phase-space distribution function

$$F(x,p) \;=\; f_{||}\;\; g_{\rm eq}$$

$f_{||}$ - non-equilibrium longitudinal part
$\rightarrow$ *free – streaming*

$g_{\rm eq}$ - equilibrium transverse part
$\rightarrow$ *2D hydrodynamic expansion*

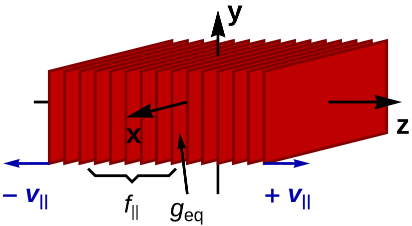
- Longitudinal part is given by the expression which implements condition $y = \eta$

$$f_{||} = n_0 \frac{\delta(y-\eta)}{m_\perp \tau} = \frac{n_0}{\tau} \delta\left(p^\mu V_\mu\right)$$

- Transversal part, in the case of local equilibrium, has the form

$$g\left(p^\mu U_\mu\right) = \frac{1}{e^{p^\mu U_\mu / T} \mp 1}$$

$$p^\mu U_\mu = m_\perp u_0 \cosh(y-\eta) - \vec{p}_\perp \cdot \vec{u}_\perp$$



Moments of the distribution function

- Particle current is defined by the first moment of the distribution function

$$N^\mu = \frac{n_0 \nu_g}{(2\pi)^2 \tau} \int \frac{d^3p}{p^0} p^\mu \delta(p \cdot V) g(p \cdot U)$$

- N^μ may be written as the linear combination

$$N^\mu = a\, V^\mu + b\, U^\mu$$

- Performing projections of the particle current on the proper four vectors we obtain coefficients which can be calculated in the local rest frame of the fluid where $U^\mu = (1, 0, 0, 0)$ and $V^\mu = (0, 0, 0, 1)$

$$N^\mu = \frac{n_0}{\tau} n_2 U^\mu$$

Moments of the distribution function

- Energy-momentum tensor is defined in the following way

$$T^{\mu\nu} = \frac{n_0 \nu_g}{(2\pi)^2 \tau} \int \frac{d^3p}{p^0} p^\mu p^\nu \delta(p \cdot V) g(p \cdot U)$$

and should have the following structure

$$T_{\mu\nu} = a' U_\mu U_\nu + b' g_{\mu\nu} + c' V_\mu V_\nu + \frac{d'}{2}(U_\mu V_\nu + U_\nu V_\mu)$$

- Simple algebra gives

$$\begin{aligned} T^{\mu}_{\ \mu} &= a' + 4b' - c' \\ T^{\mu\nu} U_\mu U_\nu &= a' + b' \\ T^{\mu\nu} V_\mu V_\nu &= -b' + c' \\ T^{\mu\nu} U_\mu V_\nu &= T^{\mu\nu} U_\nu V_\mu = -\frac{d'}{2} \end{aligned}$$

- This method, by solving system of equations, quite easily yields

$$T^{\mu\nu} = \frac{n_0}{\tau} \left[(\varepsilon_2 + P_2) U^\mu U^\nu - P_2 \ (g^{\mu\nu} + V^\mu V^\nu) \right]$$

Moments of the distribution function

- Entropy current is defined in the following way

$$S^\mu = -\frac{n_0 \nu_g}{(2\pi)^2 \tau} \int \frac{d^3p}{p^0} p^\mu \; \delta(p \cdot V) g(p \cdot U) [\; \ln[g(p \cdot U)] - 1 \;]$$

We use the Boltzmann expression to calculate the entropy of a single cluster and then the sum over clusters is performed.

- We expect

$$S^\mu = a'' V^\mu + b'' U^\mu$$

- Finally we obtain

$$S^\mu = \frac{n_0}{\tau} s_2 U^\mu$$

Conclusions

- Generally covariant formalism for hydrodynamical description of transversally thermalized matter has been provided
- Thermodynamics of two-dimensional systems has been analyzed
- Moments of the phase-space distribution function have been derived
- Hydrodynamic equations have been derived from the general form of the energy-momentum tensor