High Energy QCD and the BDS formula

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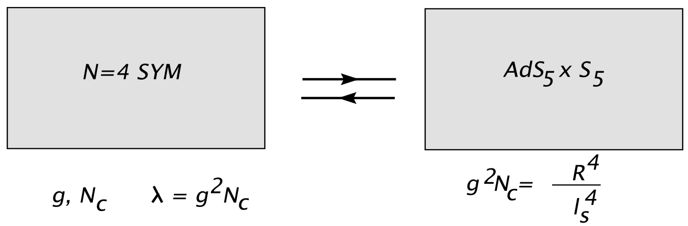
- Introduction
- High Energy Behavior in Yang Mills Theories
- Comparison with the BDS formula (Bern, Dixon, Smirnov, Phys.Rev.D 72, 085001 (2005))
- Outlook: tasks

Based upon:
JB, Lev Lipatov, Agustin Sabio Vera, arXiv:0802.2065[hep-th]; 0807.0894[hep-th];
JB, V.S.Fadin, L.N.Lipatov, in preparation.

Introduction

AdS/CFT duality offers new approach, just in the beginning of investigations

Consider N=4 supersymmetric QCD: N=4SYM = maximal symmetric, conformal symmetric, Yang Mills theory, no low energy phases, fixed coupling constant.
Duality conjecture (AdS/CFT correspondence): gauge field theory is the same theory as a string theory in 10 dimensions with $AdS_5 \otimes S_5$ geometry:



Weak coupling on the string side gives strong coupling of N=4SYM gauge theory.
Finite temperature (above all low energy phases): blackhole on the string side.

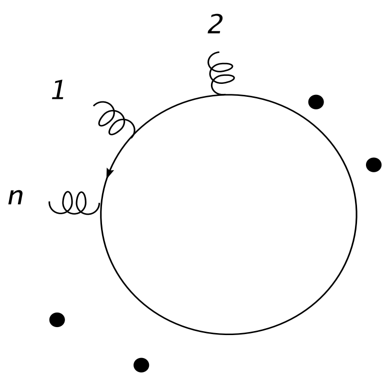
Application to high energy behavior:

- gluon scattering to all orders in g (BDS formula)
  Bern, Dixon, Smirnov; JB,Lipatov,Sabio-Vera
- mimic QCD by introducing a scale, address explicitly DIS/Pomeron in AdS/CFT
  (Polchinski, Strassler; Iancu, Mueller;Kovchegov)

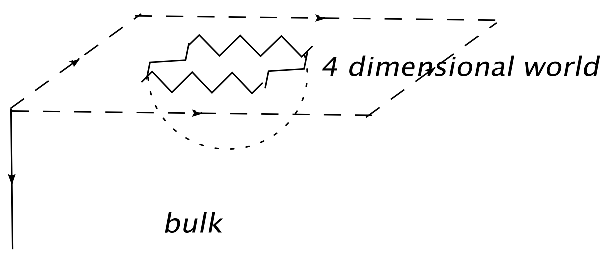
This talk: study of the BDS formula

The BDS formula:

Proposal for n point amplitude, leading order N_c, maximal helicity violating, on shell, all-order coupling constant.
If correct: major breakthrough in QFT, even if not valid for QCD.
Could help, e.g., to compute NLLO correction in QCD (e.g.BFKL kernel, anomalous dimensions.



compute amplitudes
(planar)



compute surfaces

Duality suggests exponential form of the amplitude.

Remove color factors, factor out tree amplitude, IR singular:

$$tr(T^{a1}...T^{an}) + noncycl.perm, \quad A_n = A_n^{tree} \cdot M_n(\epsilon)$$

$$\ln M_n = \sum_l a^l \Big[\Big(f^{(l)}(\epsilon) I_n(l\epsilon) + F_n(0) \Big) + C^{(l)} + E_n^{(l)}[\epsilon] \Big]$$

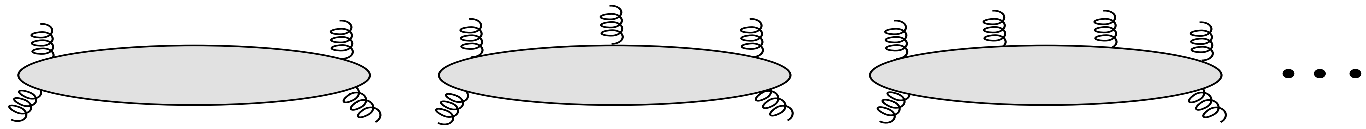
$$a = \frac{N_c \alpha}{2\pi} (4\pi e^{-\gamma})^{\epsilon}, \; d = 4 - 2\epsilon$$

Based upon: universality of IR singularities (=poles in ϵ), and 1-loop calculation.

Current status:
incorrect for $n > 5$, beyond one loop.
(Alday,Maldacena; Bern et al; Drummond et al. JB, Lipatov et al.)

Origin of the error: ongoing debate.

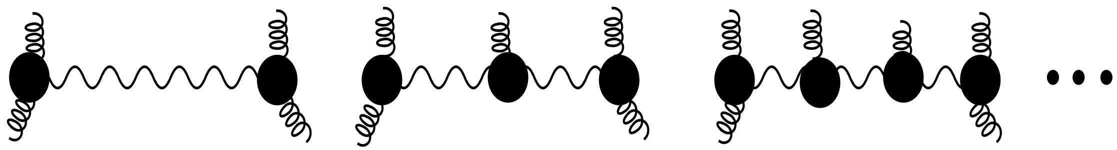
Examples of what one might expect:



Planar, MHV

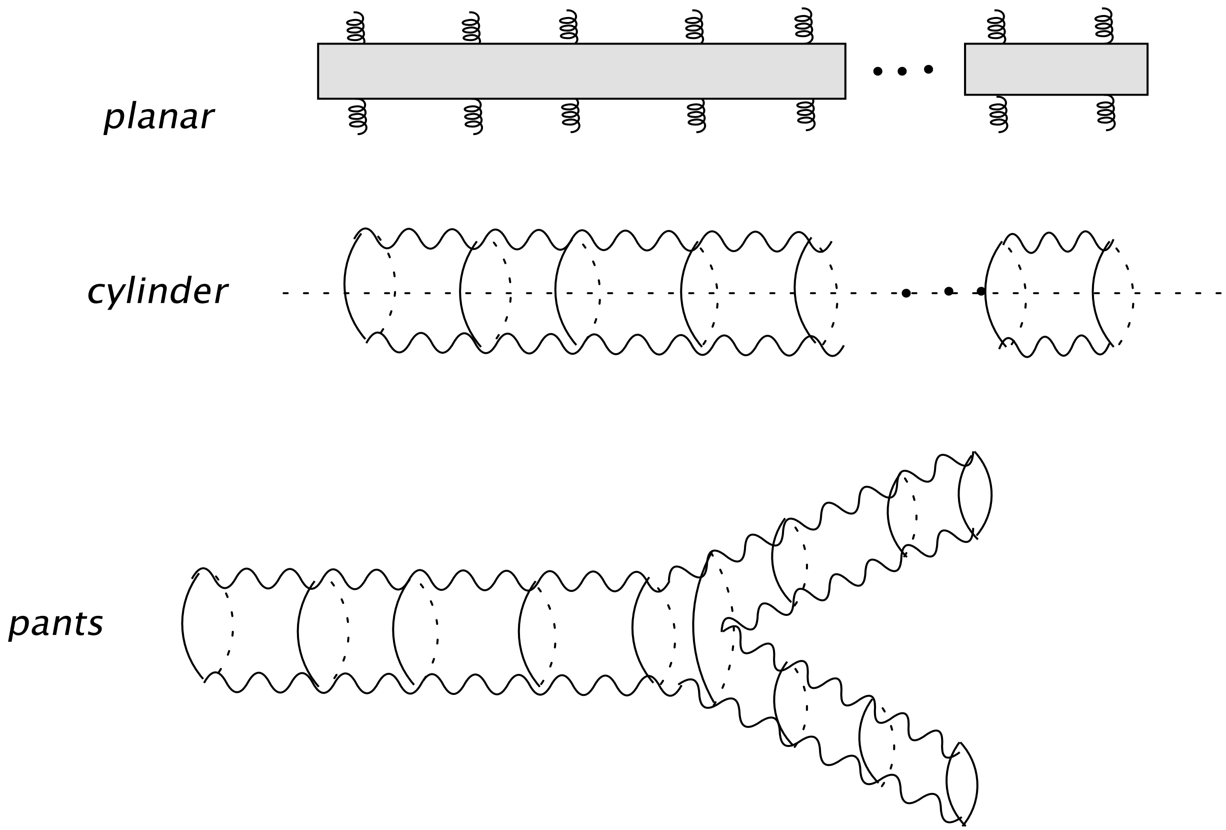
For high energies (large rapidity separation): from QCD, expect reggeized gluons

planar



$$\omega_g(t) = a\left(\frac{1}{\epsilon} - \ln(-t) + \ldots\right)$$

From planar topology to cylinder topology to pair of pants toplogy ...:



Ladder structure appears: BFKL, evolution equations. Triple Pomeron vertex

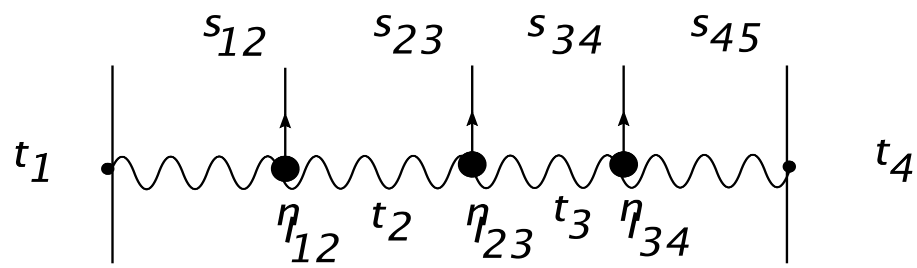
Duality: gauge theory - string side:

planar diagrams - surfaces
cylinder (R-currents) - Witten diagrams with graviton exchange ($\longrightarrow$ J.Kotanski)
pair of pants (triple Pomeron vertex) - triple graviton vertex.

This talk: how good is the BDS formula?
Compare the high energy limit with QCD (at large N_c) in LLA (no SUSY).

High energy behavior in QCD

Leading logarithmic approximation is real, e.g.:



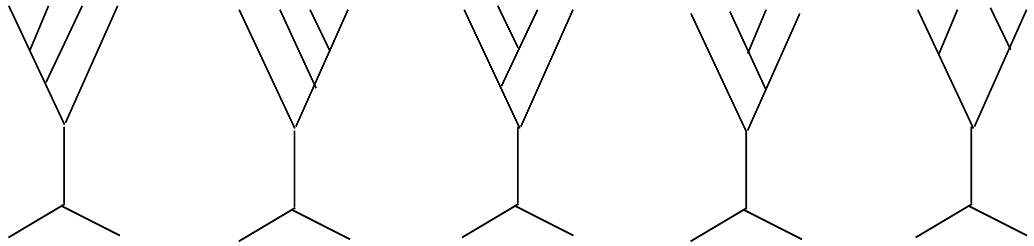
$$A_{2\to4} = 2s\,\beta^{(0)}(t_1)\,\delta_{\lambda_A,\lambda_{A'}}\frac{s_1^{\omega(t_1)}}{t_1}\,\Gamma^{(0)}(t_1,t_2,\eta_{12})\,\frac{s_2^{\omega(t_2)}}{t_2}\,\Gamma^{(0)}(t_2,t_3,\eta_{23})\,\frac{s_3^{\omega(t_3)}}{t_3}\,\beta^{(0)}(t_3)\delta_{\lambda_B,\lambda_{B'}}$$

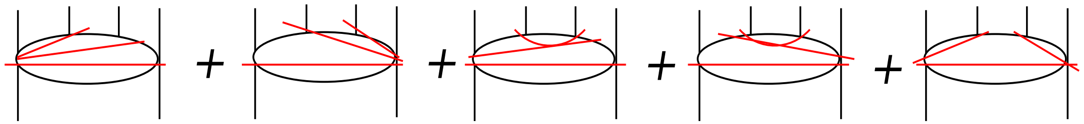
Simple factorization (exponentiation):

$$\ln M_6 = \ln\Gamma(t_1) + \omega(t_1)\ln s_1 + \ln\Gamma(t_1,t_2,\eta) + \omega(t_2)\ln s_2 + \ldots \ln\Gamma(t_3)$$

Problem of BDS resides in the imaginary parts - energy discontinuities:
independent energy variables .

Steinmann relations: 'no simultaneous discontinuities in overlapping channels'
Example: $2 \longrightarrow 4$, physical region (all energies positive)



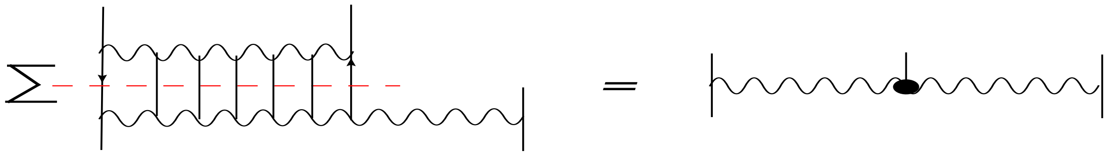
$A_6 =$ 

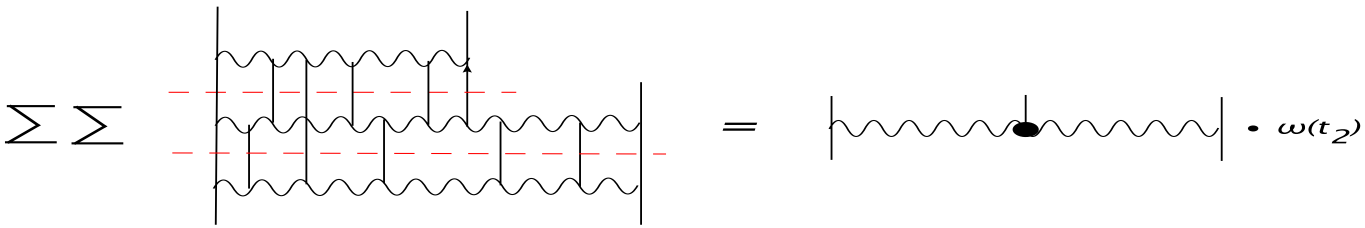
Amplitude $=$ sum of different triple discontinuties .
Analytic representation: all phases are in energy and signature factors.

Similarly $3 \longrightarrow 3$: 5 terms.

Number of terms grows: $2 \longrightarrow 5$: 14 terms, $2 \longrightarrow 6$: 42 terms etc.

Analytic representation can be used to compute all terms from (multiple) discontinuities. (JB, Nucl.Phys.B 151 and B 175; Fadin,Lipatov, Nucl.Phys.406). Example:





Bootstrap relations: known from BFKL. Hold for inelastic amplitudes.

Bootstrap relations are valid beyond leading order.

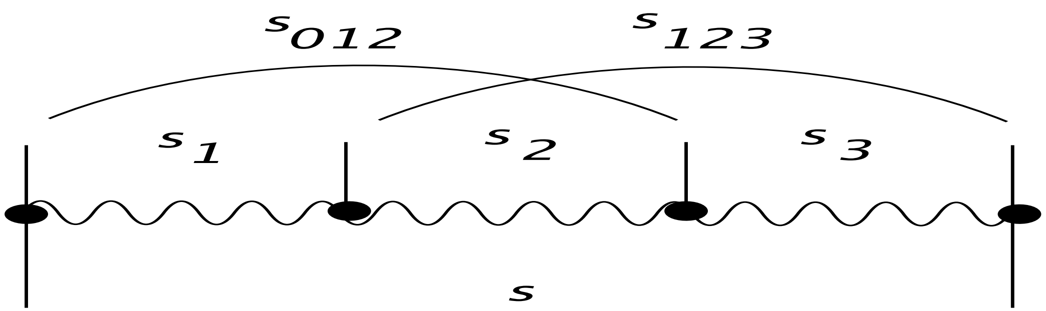
High degree of selfconsistency.

Results for QCD: five partial waves, e.g. the first term

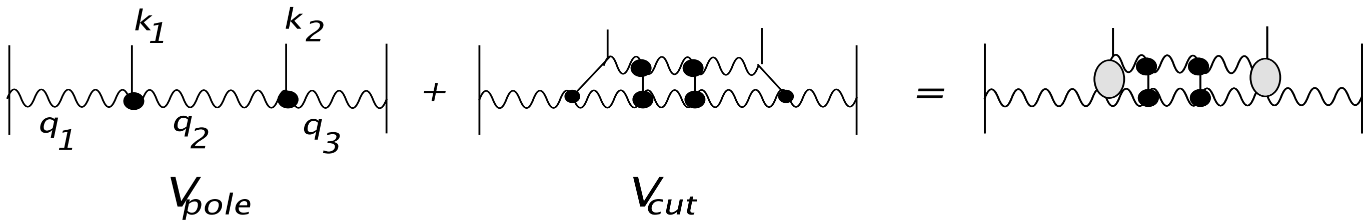
$$\frac{g^2 s}{t_1 t_2 t_3}\Big[\left(\frac{s_1}{\mu^2}\right)^{\omega(t_1)-\omega(t_2)}\left(\frac{s_{123}}{\mu^2}\right)^{\omega(t_2)-\omega(t_3)}\left(\frac{s}{\mu^2}\right)^{\omega(t_3)}\xi(t_1,t_2)\xi(t_2,t_3)\xi(t_3)\cdot$$

$$\frac{\omega(t_3)}{4}\left(\frac{a}{\epsilon}+\omega(t_1)-\omega(t_2)-a\ln\frac{\kappa_{12}}{\mu^2}\right)\cdot\left(\frac{a}{\epsilon}+\omega(t_2)-\omega(t_3)-a\ln\frac{\kappa_{23}}{\mu^2}\right)\Big]$$

Belongs to Regge pole picture:



New feature appears for terms 3 and 4:
contains not only gluon Regge pole but also Regge cut:



$$disc_{s_2} A_6 \sim s_2^{\omega(t_2)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s_2}{\mu^2}\right)^{\omega} \widetilde{f}_2(\omega),$$

$$\widetilde{f}_2(\omega) = \hat{\alpha}_\epsilon \boldsymbol{q}_2^2 \int d^{2-2\epsilon}k \, d^{2-2\epsilon}k' \, \Phi_1(\boldsymbol{k}, \boldsymbol{q}_2, \boldsymbol{q}_1) \, \widetilde{G}_\omega(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q}_2) \, \Phi_3(\boldsymbol{k}', \boldsymbol{q}_2, \boldsymbol{q}_3) \, .$$

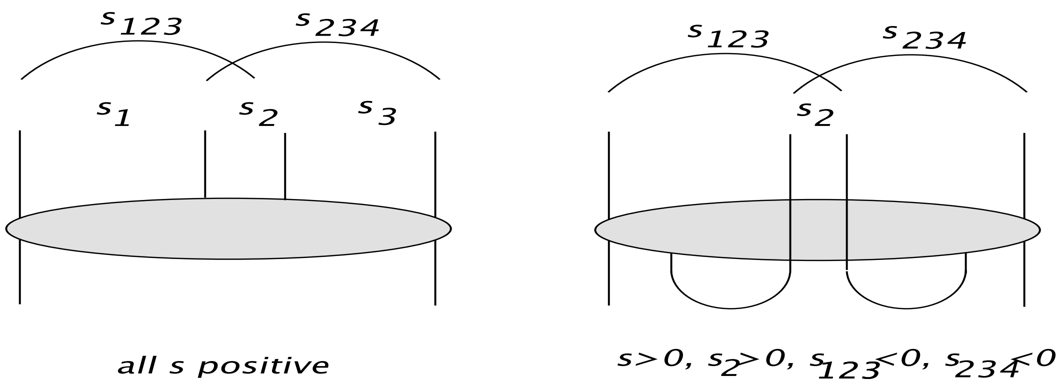
- Regge cut piece violates factorization
- Regge cut piece is present in several discontinuities, e.g. in total energy s, but not in all discontinuities.
- Regge cut piece present in all A_n with $n > 5$, e.g. $3 \to 3$.
- for $2 \longrightarrow 6$ new piece with three gluon cut: BKP, integrability

- anharmonic ratios $\Phi = \frac{s s_2}{s_{123} s_{234}}$, and $\frac{q_3 k_1}{k_2 q_1}$, $\frac{q_3^* k_1^*}{k_2^* q_1^*}$ (dual conformal invariance)

Sum the 5 different pieces and obtain the full scattering amplitudes A_n:
in planar approximation

- in physical region (where all energies are positive):
  complete factorization (in particular: real part),
  Regge cut piece cancels, simple factorizing structure is valid .
- in unphysical region (where all energies are negative):
  complete factorization
- But: in another physical region $s > 0, s_2 > 0, s_{123} < 0, s_{234} < 0$:
  Regge cut piece appears in discontinuities in s_2 s, factorization is violated .

Planar approximation: has only right hand cuts.
But still allows different physical regions:



Comparison with BDS formula

General strategy:
our analysis has been done for $\ln M$, discarding terms which vanish as $\epsilon \longrightarrow 0$.
Start from region where all invariants are negative, take multiregge limit.
Then, by analytic continuation, compare with previous result in different physical regions (all at large-N_c, MHV).

The four point amplitude:

$$\ln M_4 = 2 \ln \Gamma(t) + \omega(t) \ln(-s)/\mu^2$$

$$M_4 = \Gamma(t) \left(\frac{-s}{\mu^2} \right)^{\omega(t)} \Gamma(t)$$

- No squares of $\ln s$
- one loop expression for Γ
  and two-loop expression for $\omega(t)$ agree with explicit calculations
- exact: can also be written in 'dual' t-channel form (no high energy approximation).

The five point amplitude:

In $\ln M_5$: terms with squares of logarithms cancel. New production vertex:

$$M_{2\to 3} = \Gamma(t_1) \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma(t_2, t_1, \ln(-\kappa)) \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)} \Gamma(t_2)$$

with

$$-\kappa = \frac{(-s_1)(-s_2)}{(-s)}$$

Representation is exact.

Analytic continuation to positive energies:

$$-s \to e^{-i\pi} s, \quad \ln(-\kappa) \to \ln\kappa - i\pi, \quad \kappa = \boldsymbol{k}^2$$

Amplitude can be written in the analytic form:

$$\frac{M_{2\to 3}}{\Gamma(t_1)\Gamma(t_2)} = \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)-\omega(t_2)} \left(\frac{-s}{\mu^2}\right)^{\omega(t_2)} c_1 + \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)-\omega(t_1)} \left(\frac{-s}{\mu^2}\right)^{\omega(t_1)} c_2,$$

with real-valued functions c_1, c_2. Consistency check: the region $s_{12}, s_{23} < 0$.

The six point amplitude: $T_{2\to 4}$

In the unphysical region (all energies negative):

$$\frac{M_{2\to 4}}{\Gamma(t_1)\Gamma(t_3)} = \left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)} \Gamma(t_2, t_1, \ln -\kappa_{12}) \left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)} \Gamma(t_3, t_2, \ln -\kappa_{23}) \left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}$$

with

$$-\kappa_{12} = \frac{(-s_1)(-s_2)}{-s_{012}}, \quad -\kappa_{23} = \frac{(-s_2)(-s_3)}{-s_{123}}.$$

The same functions $\Gamma(t)$ and $\Gamma(t_1, t_2, \kappa)$ as before.

Analytic continuation: inconsistency appears .
Can be seen in several different ways:

(a) attempt to write as a sum of five terms with real-valued functions c_i (use also the other physical region: $s > 0, s_2 > 0, s_{123} < 0, s_{234} < 0$): no solution for the c_i.

(b) comparison with the earlier QCD results: in the region $s > 0, s_2 > 0, s_{123} < 0, s_{234} < 0$, one should see the Regge cut piece:

$$disc_s A_6 \sim s_2^{\omega(t_2)} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{d\omega}{2\pi i} \left(\frac{s_2}{\mu^2}\right)^{\omega} \widetilde{f}_2(\omega),$$

$$\widetilde{f}_2(\omega) = \hat{\alpha}_\epsilon \, \boldsymbol{q}_2^2 \int d^{2-2\epsilon}k \, d^{2-2\epsilon}k' \, \Phi_1(\boldsymbol{k}, \boldsymbol{q}_2, \boldsymbol{q}_1) \, \widetilde{G}_\omega(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{q}_2) \, \Phi_3(\boldsymbol{k}', \boldsymbol{q}_2, \boldsymbol{q}_3) \, .$$

$$\widetilde{f}_2 = \frac{a}{2} \left(\ln \frac{\boldsymbol{q}_1^2 \boldsymbol{q}_3^2}{(\boldsymbol{k}_1 + \boldsymbol{k}_2)^2 \mu^2} - \frac{1}{\epsilon} \right) + \frac{a^2}{2} \ln s_2 \ln \frac{|q_1 - q_3|^2 |q_2|^2}{|q_1|^2 |k_2|^2} \ln \frac{|q_1 - q_3|^2 |q_2|^2}{|q_3|^2 |k_1|^2} + \; \ldots .$$

The BDS formula yields:

$$C = \exp \left[\frac{\gamma_K(a)}{4} \, i\pi \left(\ln \frac{(-t_1)(-t_3)}{(\vec{k}_1 + \vec{k}_2)^2 \mu^2} - \frac{1}{\epsilon} \right) \right]$$

$$\approx 1 + i\pi a \left(\ln \frac{(-t_1)(-t_3)}{(\vec{k}_1 + \vec{k}_2)^2 \mu^2} - \frac{1}{\epsilon} \right) .$$

Agrees with the one loop approximation to the Regge cut piece,
but BDS cannot reproduce the full Regge cut structure.
No conflict with the infrared structure of the BDS formula

Outlook: results and tasks

What has been achieved, by comparison with explicit QCD calculations:

- BDS ok for 4 and 5 point amplitude. Regge limit is even exact.
- subtle disagreement for M_n for $n \geq 6$ beyond one loop.
- in general, expect no simple exponential form. What instead?

Can we correct the formula? Reasons for being optimistic:

- many features of the BDS formula seem already to be correct (infrared and beyond)
- structure seen in the Regge limit may not be too far from general kinematics
- experience from analyzing QCD in Regge limit: structures seen in leading log (bootstrap, unitarity properties) may survive in higher order