Higher order corrections to heavy flavor production in deep inelastic scattering

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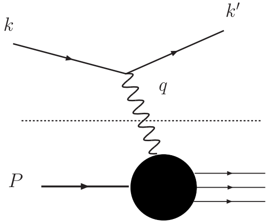
Deep–Inelastic Scattering (DIS):



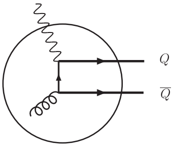
$\to L^{\mu\nu}$

$\to W_{\mu\nu}$

$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq}$ Bjørken–x

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

Heavy–flavor production: LO–process: photon–gluon fusion



Hadronic Tensor for heavy quark production via single photon exchange:

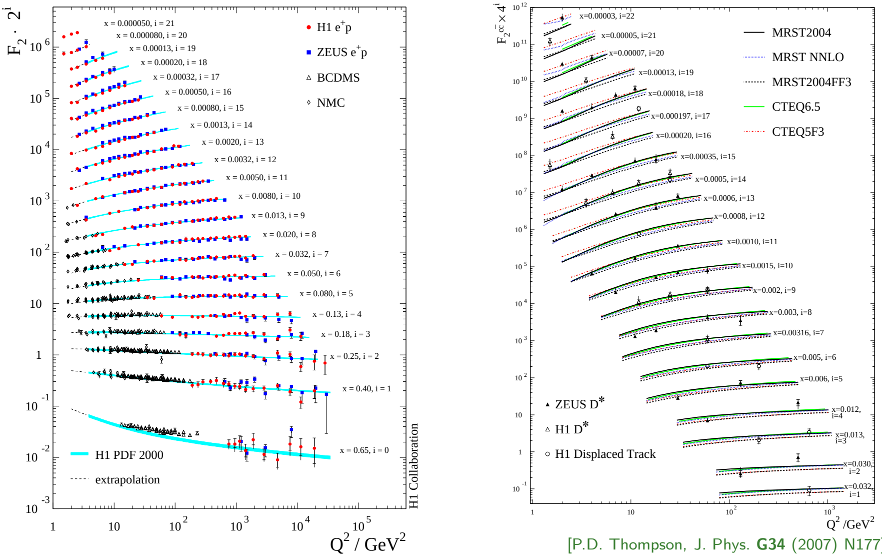
$$\begin{aligned} W_{\mu\nu}^{Q\bar{Q}}(q,P,s) &= \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle_{Q\bar{Q}} \\ &= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L^{Q\bar{Q}}(x,Q^2) \\ &\quad + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2^{Q\bar{Q}}(x,Q^2) \\ &\quad - \frac{M}{2Pq}\varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1^{Q\bar{Q}}(x,Q^2) + \left(s^\beta - \frac{sq}{Pq}p^\beta\right) g_2^{Q\bar{Q}}(x,Q^2)\right] . \end{aligned}$$

Collinear parton model $\Longrightarrow$ parton longitudinal momentum fraction: $p = zP$.

Need for the Calculation

- Heavy flavor (charm) contributions to DIS structure functions are rather large [20–40 % at lower values of x] .
- Need: Increase accuracy of the perturbative description of DIS structure functions.
- $\Longleftrightarrow$ QCD analysis and determination of $\Lambda_{\rm QCD}$ from DIS data.
- $\Longleftrightarrow$ Precise determination of the gluon and sea quark Distributions.

Goal: Calculation of the heavy flavor Wilson coefficients to higher orders for $Q^2 \geq 25\,\mathrm{GeV}^2$ [sufficient in many applications] .



[P.D. Thompson, J. Phys. **G34** (2007) N177]

Unpolarized DIS :

- LO : [Witten, 1976; Babcock & Sivers, 1978; Shifman, Vainshtein, Zakharov 1978; Leveille & Weiler, 1979]
- NLO : [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]
  asymptotic : [Buza, Matiounine, Smith, Migneron, van Neerven, 1996]

Polarized DIS :

- LO : [Watson, 1982; Glück, Reya, Vogelsang, 1991; Vogelsang, 1991]
- NLO : asymptotic: [Buza, Matiounine, Smith, van Neerven, 1997]

Mellin–Space Expressions:

[Alekhin, Blümlein, 2003].

massless RGE and Light–Cone Expansion in Bjørken–Limit $\{Q^2,\ \nu\} \ \rightarrow\ \infty,\ x$ fixed:

$$\lim_{\xi^2\to 0}\Big[J(\xi), J(0)\Big] \propto \sum_{i,N,\tau} c^N_{i,\tau}(\xi^2,\mu^2)\xi_{\mu_1}...\xi_{\mu_N} O^{\mu_1\cdots\mu_m}_{i,\tau}(0,\mu^2)$$

Operators: flavour non-singlet, singlet & pure singlet; consider leading twist-2 operators

factorization between Wilson coefficients and parton densities;

$$F_i(x,Q^2) = \sum_j \underbrace{C^j_i\left(x,\frac{Q^2}{\mu^2}\right)}_{\text{perturbative}} \otimes \underbrace{f_j(x,\mu^2)}_{\text{non-perturbative}}$$

with $$[f\otimes g](z) = \int_0^1 dz_1 \int_0^1 dz_2 \ \ \delta(z - z_1 z_2)\ f(z_1)g(z_2)\ .$$

(massless) RGE: Altarelli–Parisi (DGLAP) evolution equation for pdfs $(\mu^2 = Q^2)$:

$$\frac{d}{d\ \log Q^2} f_g(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\left\{P_{g\leftarrow q}(z)\sum_f\left[f_f(\frac{x}{z},Q) + f_{\bar f}(\frac{x}{z},Q)\right] + P_{g\leftarrow g}(z) f_g(\frac{x}{z},Q)\right\}$$

$P_{i\leftarrow j}$ are the splitting functions.

Heavy quark contribution: heavy quark Wilson coefficient, $H^{\rm S,NS}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)$

In limit $Q^2 \gg m_Q^2$:

massive RGE, derivative $m^2\partial/\partial m$: all terms but power corrections, calculable through partonic operator matrix elements, $\langle i|A_l|j\rangle$, which are process independent objects!

$$H^{\rm S,NS}_{(2,L),i}\left(\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right) = \underbrace{A^{\rm S,NS}_{k,i}\left(\frac{m^2}{\mu^2}\right)}_{\text{massive OMEs}} \otimes \underbrace{C^{\rm S,NS}_{(2,L),k}\left(\frac{Q^2}{\mu^2}\right)}_{\text{light–Wilson coefficients}} .$$

holds for polarized and unpolarized case.

OMEs obey expansion

$$A^{\rm S,NS}_{k,i}\left(\frac{m^2}{\mu^2}\right) = \langle i|O^{\rm S,NS}_k|i\rangle = \delta_{k,i} + \sum_{l=1}^{\infty} a_s^l A^{{\rm S,NS},(l)}_{k,i}\left(\frac{m^2}{\mu^2}\right), \quad i = q, g$$

[Buza, Matiounine, Migneron, Smith, van Neerven, 1996; Buza, Matiounine, Smith, van Neerven, 1997.]

Expansion up to $O(\alpha_s^2)$ of unpolarized Heavy Flavor Wilson Coefficient H_2:

$$
\begin{aligned}
H_{2,g}^{S}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= a_s\left[A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right)+\widehat{C}_{2,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right)\right] \\
&+ a_s^2\left[A_{Qg}^{(2)}\left(\frac{m^2}{\mu^2}\right)+A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right)\otimes C_{2,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right)+\widehat{C}_{2,g}^{(2)}\left(\frac{Q^2}{\mu^2}\right)\right], \\
H_{2,q}^{PS}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= a_s^2\left[A_{Qq}^{\mathrm{PS},(2)}\left(\frac{m^2}{\mu^2}\right)+\widehat{C}_{2,q}^{\mathrm{PS},(2)}\left(\frac{Q^2}{\mu^2}\right)\right], \\
H_{2,q}^{\mathrm{NS}}\left(\frac{Q^2}{m^2},\frac{m^2}{\mu^2}\right) &= a_s^2\left[A_{qq,Q}^{\mathrm{NS},(2)}\left(\frac{m^2}{\mu^2}\right)+\widehat{C}_{2,q}^{\mathrm{NS},(2)}\left(\frac{Q^2}{\mu^2}\right)\right].
\end{aligned}
$$

- Polarized and longitudinal Heavy Wilson coefficients obey similar expansion.
- For H_L, $O(a_s^3)$ contributions have been derived recently.
  [J. Blümlein, A. De Freitas, W. van Neerven, S. Klein, 2006].

The universal massive operator matrix elements have the same structure in the polarized and unpolarized case $\left[F_2 \text{ vs. } g_1\right]$. Up to $O(a_s^2)$ they are given by:

$$A_{Qg}^{(1)} = -\frac{1}{2}\widehat{P}_{qg}^{(0)}\ln\left(\frac{m^2}{\mu^2}\right)$$

$$A_{Qg}^{(2)} = \frac{1}{8}\left\{\widehat{P}_{qg}^{(0)}\otimes\left[P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0\right]\right\}\ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}_{qg}^{(1)}\ln\left(\frac{m^2}{\mu^2}\right) + \overline{a}_{Qg}^{(1)}\otimes\left[P_{qq}^{(0)} - P_{gg}^{(0)} + 2\beta_0\right] + a_{Qg}^{(2)}$$

$$A_{Qq}^{\mathrm{PS},(2)} = -\frac{1}{8}\widehat{P}_{qg}^{(0)}\otimes P_{gq}^{(0)}\ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}_{qq}^{\mathrm{PS},(1)}\ln\left(\frac{m^2}{\mu^2}\right) + a_{Qq}^{\mathrm{PS},(2)} + \overline{a}_{Qg}^{(1)}\otimes P_{gq}^{(0)}$$

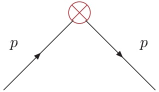
$$A_{qq,Q}^{\mathrm{NS},(2)} = -\frac{\beta_{0,Q}}{4}P_{qq}^{(0)}\ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\widehat{P}_{qq}^{\mathrm{NS},(1)}\ln\left(\frac{m^2}{\mu^2}\right) + a_{qq,Q}^{\mathrm{NS},(2)} + \frac{1}{4}\beta_{0,Q}\zeta_2 P_{qq}^{(0)}\ .$$

- $\overline{a}_{Qg}^{(1)}$: $O(\varepsilon)$ of the OME $A_{Qg}^{(1)}$ in the $\overline{\mathrm{MS}}$–scheme. Enters $A_{Qg}^{(2)}$ through renormalization.
- Renormalization of $A_{Qg}^{(3)}$: $\overline{a}_{Qg}^{(2)}$ is needed ($O(\varepsilon)$–term of $A_{Qg}^{(2)}$).

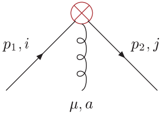
Operator Insertions in Light–Cone Expansion

E.g. singlet heavy quark operator:

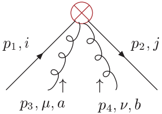
$$O_Q^{\mu_1 \dots \mu_N}(z) = \frac{1}{2} i^{N-1} \, \overline{q}(z) \gamma^{\{\mu_1} D^{\mu_2} \dots D^{\mu_N\}} q(z) - \text{trace terms} \; .$$



$$\not\Delta \gamma_\pm (\Delta \cdot p)^{N-1} \; ,$$



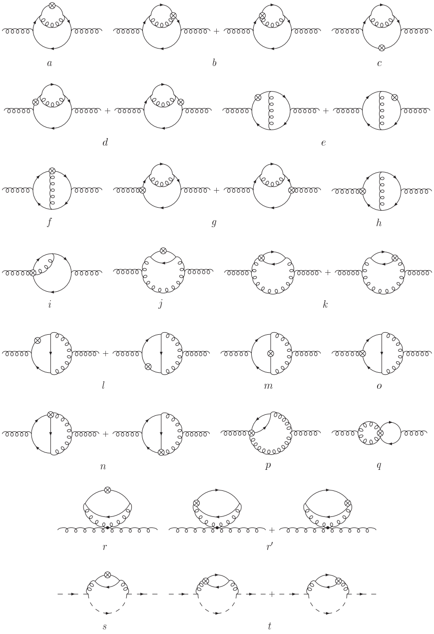
$$g t^a_{ji} \Delta^\mu \not\Delta \gamma_\pm \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (\Delta \cdot p_2)^{N-j-2} \; ,$$



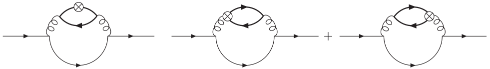
$$g^2 \Delta^\mu \Delta^\nu \not\Delta \gamma_\pm \sum_{0 \le j < l}^{N-2} \Big[(\Delta p_1)^{N-l-2} (\Delta p_1 + \Delta p_4)^{l-j-1} (\Delta p_2)^j (t^a t^b)_{ji} + (\Delta p_1)^{N-l-2} (\Delta p_1 + \Delta p_3)^{l-j-1} (\Delta p_2)^j (t^b t^a)_{ji} \Big] \; ,$$

$$\gamma_+ = 1 \; , \quad \gamma_- = \gamma_5 \; .$$

Diagrams contributing to the gluonic OME $\hat{A}_{Qg}^{(2)}$:

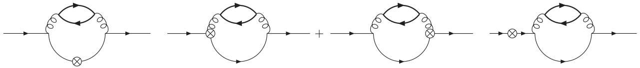


Pure singlet:



a b

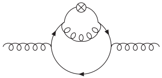
Non singlet:



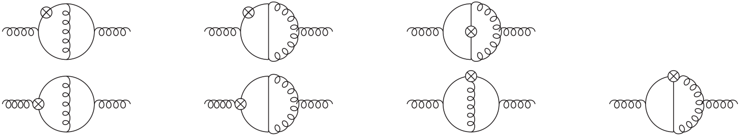
a b c

$\Rightarrow$ In general, two types of diagrams:

- 2-loop diagrams with 2-point 1-loop insertions



- Genuine Two-loop:



Calculation in Mellin-space:

$$\Rightarrow \quad F(N) = \mathbf{M}[f,N] = \int_0^1 x^{N-1}\, f(x)\, dx$$

Convolution:

$$[f \otimes g](x) = \int_0^1 dx_1 \int_0^1 dx_2 \;\; \delta(x - x_1 x_2)\; f(x_1) g(x_2)\ ,$$

⇒ Product:

$$\mathbf{M}[f \otimes g, N] = \mathbf{M}[f,N]\,\mathbf{M}[g,N] \;\; = F(N)\; G(N).$$

$$F_2^{Q\overline{Q}} \quad = \quad \sum_{k=1}^{n_f} e_k^2 \left[f_{k-\overline{k}}(N,\mu^2) H_{2,q}^{NS}\left(N, \frac{Q^2}{m^2}, \frac{Q^2}{\mu^2}\right)\right]$$
$$+ e_Q^2 \left[\Sigma(N,\mu^2) H_{2,q}^{PS}\left(N, \frac{Q^2}{m^2}, \frac{Q^2}{\mu^2}\right) + G(N,\mu^2) H_{2,q}^{S}\left(N, \frac{Q^2}{m^2}, \frac{Q^2}{\mu^2}\right)\right]$$

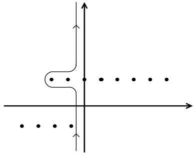
light-quark densities:

$$f_{k-\overline{k}}(N,\mu^2) \quad = \quad f_k(N,\mu^2) - f_{\overline{k}}(N,\mu^2),$$
$$\Sigma(N,\mu^2) \quad = \quad \sum_{k=1}^{n_f} f_{k+\overline{k}}(N,\mu^2).$$

Our calculation:

- use of Mellin-Barnes integrals

$$\frac{1}{(A+B)^{\nu}} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\sigma A^{\sigma} B^{-\nu-\sigma} \frac{\Gamma(-\sigma)\Gamma(\nu+\sigma)}{\Gamma(\nu)}$$



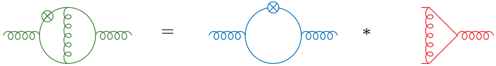
$\rightsquigarrow$ numerical check & some analytic results

- use of hypergeometric functions for general analytic results

$${}_PF_Q\left[\begin{array}{c}(a_1)...(a_P)\\(b_1)...(b_Q)\end{array};z\right] = \sum_{i=0}^{\infty}\frac{(a_1)_i...(a_P)_i}{(b_1)_i...(b_Q)_i}\frac{z^i}{\Gamma(i+1)}, \quad (c)_i = \frac{\Gamma(c+i)}{\Gamma(c)}.$$

Calculating scalar Feynman diagrams by Mellin-Barnes integrals:

[I.B., S. Weinzierl, 2003 (massless case); I.B., J. Blümlein and S. Klein, 2006]

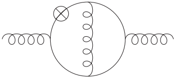


$$I_{e,\nu_1} = \frac{(\Delta p)^{N-1}}{(4\pi)^D (2\pi i)^2} \frac{(m^2)^{D-\nu_{12345}}(-1)^{\nu_{12345}+1}}{\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_5)\Gamma(D-\nu_{235})} \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} d\sigma \int_{\gamma_2-i\infty}^{\gamma_2+i\infty} d\tau\, \Gamma(-\sigma)\Gamma(\nu_3+\sigma)$$
$$\times \frac{\Gamma(-\sigma+\nu_4+N-1)}{\Gamma(-\sigma+\nu_4)} \Gamma(-\tau)\Gamma(\nu_2+\tau) \frac{\Gamma(\sigma+\tau+\nu_{235}-D/2)\Gamma(\sigma+\tau+\nu_5)}{\Gamma(\sigma+\tau+\nu_{23})}$$
$$\times \Gamma(-\sigma-\tau+D-\nu_{23}-2\nu_5) \frac{\Gamma(-\sigma-\tau+\nu_{14}-D/2)}{\Gamma(-\sigma-\tau+\nu_{14}+N-1)} ,$$

N	2	3	4	5
$\mathrm{I}_{e,1}$	+0.49999	+0.31018	+0.21527	+0.16007
$\mathrm{I}_{e,2}$	−0.09028	−0.04398	−0.02519	−0.01596

[Mathematica package MB, M. Czakon, 2006]

Hypergeometric functions: Example, scalar Diagram e:



$p^2 = 0$

$$I_{e,1} := \iint \frac{d^D q \, d^D k}{(2\pi)^{2D}} \frac{(\Delta q)^{N-1}}{[q^2 - m^2]^a [(q-p)^2 - m^2][k^2 - m^2][(k-p)^2 - m^2][(k-q)^2]}$$

$$I_{e,1} := \frac{(\Delta p)^{N-1} \Gamma(1-\varepsilon)}{N(N+1)(4\pi)^{4+\varepsilon}(m^2)^{1-\varepsilon}} \int_0^1 dz \int_0^1 dw \, \frac{w^{-1-\varepsilon/2}(1-z)^{\varepsilon/2} z^{-\varepsilon/2}}{(z+w-wz)^{1-\varepsilon}} \left[1 - w^{N+1} - (1-w)^{N+1}\right],$$

$$\int_0^1 dz \int_0^1 dw \, \frac{(1-w)^a w^b z^c (1-z)^d}{(z+w-zw)^e} = B(d+1, c+1)\, B(a+1, b+1)\; {}_3F_2\left[\begin{matrix} e, d+1, a+1 \\ 2+c+d, 2+a+b \end{matrix}; 1\right]$$

$$I_{e,1} = \frac{S_\varepsilon^2}{(4\pi)^4 (m^2)^{1-\varepsilon}} \frac{(\Delta p)^{N-1}}{N(N+1)} \exp\left\{\sum_{i=2}^{\infty} \zeta_i \frac{\varepsilon^i}{i}\right\} \Bigg\{$$

$$B(\varepsilon/2+1, 1-\varepsilon/2) B(1, -\varepsilon/2)\, {}_3F_2 \left[\begin{matrix} 1-\varepsilon, 1, 1+\varepsilon/2 \\ 2, 1-\varepsilon/2 \end{matrix} ; 1 \right]$$

$$- B(\varepsilon/2+1, 1-\varepsilon/2) B(1, N+1-\varepsilon/2)\, {}_3F_2 \left[\begin{matrix} 1-\varepsilon, 1, 1+\varepsilon/2 \\ 2, N+2-\varepsilon/2 \end{matrix} ; 1 \right]$$

$$- B(\varepsilon/2+1, 1-\varepsilon/2) B(N+2, -\varepsilon/2)\, {}_3F_2 \left[\begin{matrix} 1-\varepsilon, N+2, 1+\varepsilon/2 \\ 2, N+2-\varepsilon/2 \end{matrix} ; 1 \right] \Bigg\}$$

$$\Psi(x) = \frac{1}{\Gamma(x)} \frac{d}{dx} \Gamma(x) \qquad \Psi(N+1) = S_1(N) - \gamma\ , \quad N \in \mathbb{N}$$

$$\Psi^k(N+1) = (-1)^k \Gamma(k+1) \left[S_{k+1}(N) - \zeta(k+1) \right]\ , \quad k > 1$$

harmonic sums: [Vermaseren, 1999; Blümlein, Kurth, 1999]

$$S_{a_1,\ldots,a_m}(N) = \sum_{n_1=1}^{N} \sum_{n_2=1}^{n_1} \cdots \sum_{n_m=1}^{n_{m-1}} \frac{(\mathrm{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\mathrm{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \cdots \frac{(\mathrm{sign}(a_m))^{n_m}}{n_m^{|a_m|}}$$

$$I_{e,1} = \frac{-S_\varepsilon^2}{(4\pi)^4(m^2)^{1-\varepsilon}} \frac{(\Delta p)^{N-1}}{N(N+1)} \sum_{s=1}^{\infty} \left\{ -\frac{1}{s^2} + \frac{S_1(s)}{s} - \frac{S_1(N+s)}{s} + \frac{B(N+1,s)}{s} \right\} + O(\varepsilon)$$

$$= \frac{S_\varepsilon^2}{(4\pi)^4(m^2)^{1-\varepsilon}} (\Delta p)^{N-1} \left\{ \frac{S_1^2(N) + 3S_2(N)}{2N(N+1)} + \frac{S_1^3(N) + 3S_1(N)S_2(N) + 8S_3(N)}{12N(N+1)} \varepsilon \right\}$$

More complicated sums → partly solved with C. Schneider's mathematica package SIGMA [I.B., J. Blümlein, S. Klein, C. Schneider, 2007].

Unpolarized case, examples for individual diagrams – numeric:

Diagram	N	$1/\varepsilon^2$	$1/\varepsilon$	1	ε	ε^2
b	2	−8	4.66666	−8.82690	2.47728	−5.69523
	6	−7.73333	0.81936	−8.89777	−1.84111	−7.25674
c	2	−8	39.6	−7.23431	34.66217	6.52891
	6	−2.66666	16.53968	−2.68048	14.25224	2.77564
d	2	−8	7.86666	−6.34542	4.71236	−2.18586
	6	−2.66666	−0.69523	−2.60657	−1.74990	−2.37611
e	2	8.88889	−11.2593	9.82824	−12.8921	2.39145
	6	2.93878	−4.24257	3.39094	−4.3892	0.826978
f	2	5.33333	−9.77777	18.34139	−2.52360	16.20210
	6	3.31428	−6.87289	12.25672	−1.63790	10.86956
g	2	2.66666	−9.55555	4.59662	−8.92015	1.07313
	6	0.57142	−2.00204	1.04814	−1.89142	0.32219

$$
\begin{aligned}
&A_e^{Qg} \\
&= T_R\left[C_F-\frac{C_A}{2}\right]\Biggl\{\frac{1}{\varepsilon^2}\frac{16(N+3)}{(N+1)^2}+\frac{1}{\varepsilon}\left[-\frac{8(N+2)}{N(N+1)}S_1(N)-8\frac{3N^3+9N^2+12N+4}{N(N+1)^3(N+2)}\right] \\
&+\Biggl[-2\frac{9N^4+40N^3+71N^2-12N-36}{N(N+1)^2(N+2)(N+3)}S_2(N)-2\frac{N^3-N^2-8N-36}{N(N+1)(N+2)(N+3)}S_1^2(N) \\
&+4\frac{(N+3)}{(N+1)^2}\zeta_2+4\frac{P_{e4}}{N^2(N+1)^2(N+2)(N+3)}S_1(N)+\frac{4P_4(N)}{N^2(N+1)^4(N+2)^2(N+3)}\Biggr] \\
&+\varepsilon\Biggl[-2\frac{N+2}{N(N+1)}\Bigl(2S_{2,1}(N)+S_1(N)\zeta_2\Bigr)-\frac{2}{3}\frac{13N^4+60N^3+111N^2+4N-36}{N(N+1)^2(N+2)(N+3)}S_3(N) \\
&-\frac{1}{3}\frac{N^3-N^2-8N-36}{N(N+1)(N+2)(N+3)}\Bigl(3S_2(N)S_1(N)+S_1^3(N)\Bigr)-2\frac{3N^3+9N^2+12N+4}{N(N+1)^3(N+2)}\zeta_2 \\
&+\frac{P_{e1}}{N^2(N+1)^3(N+2)(N+3)}S_2(N)+\frac{4N^5+11N^4+15N^3-86N^2-92N-24}{N^2(N+1)^2(N+2)(N+3)}S_1^2(N) \\
&-2\frac{P_{e2}}{N^2(N+1)^3(N+2)^2(N+3)}S_1(N)-2\frac{P_{e3}}{N^3(N+1)^5(N+2)^3(N+3)}+\frac{4}{3}\frac{N+3}{(N+1)^2}\zeta_3\Biggr]\Biggr\}
\end{aligned}
$$

Results to order O(1): [I.B., J. Blümlein, S. Klein, 2006 & 2007]

Unpolarized case, Singlet O(ε)

$$
\begin{aligned}
\overline{a}_{Qg}^{(2)} = T_R C_F \Bigg\{ & \frac{2}{3}\frac{(N^2+N+2)(3N^2+3N+2)}{N^2(N+1)^2(N+2)}\zeta_3 + \frac{P_1}{N^3(N+1)^3(N+2)}S_2 + \frac{N^4-5N^3-32N^2-18N-4}{N^2(N+1)^2(N+2)}S_1^2 \\
& + \frac{N^2+N+2}{N(N+1)(N+2)}\Big(16S_{2,1,1} - 8S_{3,1} - 8S_{2,1}S_1 + 3S_4 - \frac{4}{3}S_3S_1 - \frac{1}{2}S_2^2 - S_2S_1^2 - \frac{1}{6}S_1^4 + 2\zeta_2S_2 - 2\zeta_2S_1^2 - \frac{8}{3}\zeta_3S_1\Big) \\
& - 8\frac{N^2-3N-2}{N^2(N+1)(N+2)}S_{2,1} + \frac{2}{3}\frac{3N+2}{N^2(N+2)}S_1^3 + \frac{2}{3}\frac{3N^4+48N^3+43N^2-22N-8}{N^2(N+1)^2(N+2)}S_3 + 2\frac{3N+2}{N^2(N+2)}S_2S_1 + 4\frac{S_1}{N^2}\zeta_2 \\
& + \frac{N^5+N^4-8N^3-5N^2-3N-2}{N^3(N+1)^3}\zeta_2 - 2\frac{2N^5-2N^4-11N^3-19N^2-44N-12}{N^2(N+1)^3(N+2)}S_1 + \frac{P_2}{N^5(N+1)^5(N+2)}\Bigg\} \\
+ T_R C_A \Bigg\{ & \frac{N^2+N+2}{N(N+1)(N+2)}\Big(16S_{-2,1,1} - 4S_{2,1,1} - 8S_{-3,1} - 8S_{-2,2} - 4S_{3,1} - \frac{2}{3}\beta''' + 9S_4 - 16S_{-2,1}S_1 \\
& + \frac{40}{3}S_1S_3 + 4\beta''S_1 - 8\beta'S_2 + \frac{1}{2}S_2^2 - 8\beta'S_1^2 + 5S_1^2S_2 + \frac{1}{6}S_1^4 - \frac{10}{3}S_1\zeta_3 - 2S_2\zeta_2 - 2S_1^2\zeta_2 - 4\beta'\zeta_2 - \frac{17}{5}\zeta_2^2\Big) \\
& + \frac{4(N^2-N-4)}{(N+1)^2(N+2)^2}\Big(-4S_{-2,1} + \beta'' - 4\beta'S_1\Big) - \frac{2}{3}\frac{N^3+8N^2+11N+2}{N(N+1)^2(N+2)^2}S_1^3 + 8\frac{N^4+2N^3+7N^2+22N+20}{(N+1)^3(N+2)^3}\beta' \\
& + 2\frac{3N^3-12N^2-27N-2}{N(N+1)^2(N+2)^2}S_2S_1 - \frac{16}{3}\frac{N^5+10N^4+9N^3+3N^2+7N+6}{(N-1)N^2(N+1)^2(N+2)^2}S_3 - 8\frac{N^2+N-1}{(N+1)^2(N+2)^2}\zeta_2S_1 \\
& - \frac{2}{3}\frac{9N^5-10N^4-11N^3+68N^2+24N+16}{(N-1)N^2(N+1)^2(N+2)^2}\zeta_3 - \frac{P_3}{(N-1)N^3(N+1)^3(N+2)^3}S_2 - \frac{2P_4}{(N-1)N^3(N+1)^3(N+2)^2}\zeta_2 \\
& - \frac{P_5}{N(N+1)^3(N+2)^3}S_1^2 + \frac{2P_6}{N(N+1)^4(N+2)^4}S_1 - \frac{2P_7}{(N-1)N^5(N+1)^5(N+2)^5}\Bigg\}.
\end{aligned}
$$

Unpolarized case, pure-singlet and non-singlet, $O(\varepsilon)$

$$\overline{a}_{Qq}^{\mathrm{PS},(2)} = C_F T_R \Bigg\{ -2 \frac{(5N^3+7N^2+4N+4)(N^2+5N+2)}{(N-1)N^3(N+1)^3(N+2)^2}\Big(2S_2(N)+\zeta_2\Big) - \frac{4}{3}\frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)}\Big(3S_3(N)+\zeta_3\Big) + 2\frac{P_9}{(N-1)N^5(N+1)^5(N+2)^4} \Bigg\} .$$

$$\overline{a}_{qq,Q}^{\mathrm{NS},(2)} = C_F T_R \Bigg\{ \frac{4}{3}S_4(N) + \frac{4}{3}S_2(N)\zeta_2 - \frac{8}{9}S_1(N)\zeta_3 - \frac{20}{9}S_3(N) - \frac{20}{9}S_1(N)\zeta_2 + 2\frac{3N^2+3N+2}{9N(N+1)}\zeta_3 + \frac{112}{27}S_2(N) + \frac{3N^4+6N^3+47N^2+20N-12}{18N^2(N+1)^2}\zeta_2 - \frac{656}{81}S_1(N) + \frac{P_8}{648N^4(N+1)^4} \Bigg\} .$$

First Calculation to O(α_S^2): [Buza, Matiounine, Smith, Migneron, van Neerven, 1996]

- ⇝ Integration-by-parts method
- ⇝ direct integration of individual Feynman-parameter integrals in z-space

combinations of Nielsen integrals: $S_{p,n}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 \frac{dz}{z} \ln^{n-1}(z) \ln^p(1-zx)$

$\delta(1-x)$	1	$\ln(x)$	$\ln^2(x)$	$\ln^3(x)$	$\ln(1-x)$
$\ln^2(1-x)$	$\ln^3(1-x)$	$\ln(x)\ln(1-x)$	$\ln(x)\ln^2(1-x)$	$\ln^2(x)\ln(1-x)$	$\ln(1+x)$
$\ln(x)\ln(1+x)$	$\ln^2(x)\ln(1+x)$	$\mathrm{Li}_2(1-x)$	$\ln(x)\mathrm{Li}_2(1-x)$	$\ln(1-x)\mathrm{Li}_2(1-x)$	$\mathrm{Li}_3(1-x)$
$S_{1,2}(1-x)$	$S_{1,2}(-x)$	$\frac{1}{1-x}$	$\frac{1}{1+x}$	$\frac{\ln(x)}{1-x}$	$\frac{\ln^2(x)}{1-x}$
$\frac{\ln^3(x)}{1-x}$	$\frac{\ln(x)}{1+x}$	$\frac{\ln^2(x)}{1+x}$	$\frac{\ln^3(x)}{1+x}$	$\frac{\ln(1+x)}{1+x}$	$\frac{\ln(x)\ln(1+x)}{1+x}$
$\frac{\ln(x)\ln^2(1+x)}{1+x}$	$\frac{\ln^2(x)\ln(1+x)}{1+x}$	$\frac{\ln(x)\ln(1-x)}{1-x}$	$\frac{\ln(x)\ln^2(1-x)}{1-x}$	$\frac{\ln(1-x)\mathrm{Li}_2(x)}{1-x}$	$\frac{\mathrm{Li}_2(1-x)}{1-x}$
$\frac{\ln(x)\mathrm{Li}_2(1-x)}{1-x}$	$\frac{\ln(x)\mathrm{Li}_2(1-x)}{1+x}$	$\frac{\ln(1+x)\mathrm{Li}_2(-x)}{1+x}$	$\ln(1+x)\mathrm{Li}_2(-x)$	$\mathrm{Li}_2(-x)$	$\frac{\mathrm{Li}_2(-x)}{1+x}$
$\frac{\ln(x)\mathrm{Li}_2(-x)}{1+x}$	$\frac{\mathrm{Li}_3(1-x)}{1-x}$	$\frac{\mathrm{Li}_3(-x)}{1+x}$	$\frac{S_{1,2}(1-x)}{1-x}$	$\frac{S_{1,2}(1-x)}{1+x}$	$\frac{S_{1,2}(-x)}{1+x}$

Complexity of the results in Mellin space, unpolarized case to order $O(\varepsilon)$:

Diag	S_1	S_2	S_3	S_4	S_{-2}	S_{-3}	S_{-4}	$S_{2,1}$	$S_{-2,1}$	$S_{-2,2}$	$S_{3,1}$	$S_{-3,1}$	$S_{2,1,1}$	$S_{-2,1,1}$
a		++	+											
b	++	++	++	+				++			+		+	
c		++	+											
d	++	++	+					+						
e	++	++	+					+						
f	++	++	++	+				++					+	
g	++	++	+					+						
h	++	++	+					+						
i	++	++	++	+	++	++	+	++	++	+	+	+	+	+
j		++	+											
k		++	+											
l	++	++	++	+				++			+		+	
m		++	+											
n	++	++	++	+	++	++	+	++	++	+	+	+	+	+
o	++	++	++	+				++			+		+	
p	++	++	++	+				++			+		+	
s		++	+											
t		++	+											
PS_a		++	+											
PS_b		++	+											
NS_a														
NS_b	++	++	++	+										
Σ	++	++	++	+	++	++	+	+	++	+	+	+	+	+

$+ = O(1)$ $+ = O(\varepsilon)$

unpolarized case and polarized case: van Neerven et al. to O(1):
unpolarized: 48 basic functions polarized: 24 basic functions

O(1): $\{S_1,\ S_2,\ S_3,\ S_{-2}, S_{-3}\}$, $S_{-2,1} \Longrightarrow$ 2 basic objects.

O(ε): $\{S_1,\ S_2,\ S_3,\ S_4,\ S_{-2}, S_{-3}, S_{-4}\}$, $S_{2,1}$, $S_{-2,1}$, $S_{-3,1}$, $S_{2,1,1}$, $S_{-2,1,1}$

$S_{-2,2}$ depends on $S_{-2,1}$, $S_{-3,1}$

$S_{3,1}$ depends on $S_{2,1}$

$\Longrightarrow$ 6 basic objects

$$\beta(N+1) = (-1)^N [S_{-1}(N) + \ln(2)] \ ,$$

$$\beta^{(k)}(N+1) = \Gamma(k+1)(-1)^{N+k}[S_{-k-1}(N) + (1-2^{-k})\zeta_{k+1}] \ , \ k \geq 2 \ ,$$

$$\mathbf{M}\left[\frac{\mathrm{Li}_2(x)}{1+x}\right](N+1) - \zeta_2 \beta(N+1) = (-1)^{N+1}[S_{-2,1}(N) + \frac{5}{8}\zeta_3]$$

harmonic sums with index $\{-1\}$ cancel (holds even for each diagram)

[cf. Blümlein, 2004; Blümlein and Ravindran, 2005; Blümlein and Moch, in preparation].

Calculation of quark–mass effects in QCD Wilson–coefficients in asymptotic regime $Q^2 \gg m^2$

- Calculation in Mellin space
  $\longrightarrow$ essential for simplification of calculation
- Use of Mellin–Barnes integrals (mainly numerical checks) and generalized hypergeometric functions
- Results in term of nested harmonic sums
  $\longrightarrow$ use of algebraic relations of harmonic sums for simplification of results
- Calculation of the constant term of the Operator Matrix Elements
  $\longrightarrow$ full agreement with results of van Neerven et al.
- New: Calculation of the $O(\varepsilon)$ term of the two-loop OMEs complete, necessary for the calculation of the Heavy Wilson coefficients up to $O(\alpha_s^3)$