Spin amplitudes and gauge invariance for a QCD Monte Carlo[a]

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[a]This work is partly supported by the EU grant mTkd-CT-2004-510126 in partnership with the CERN Physics Department and by the Polish Ministry of Scientific Research and Information Technology grant No 620/E-77/6.PRUE/DIE 188/ 2005-2008.

Motivation

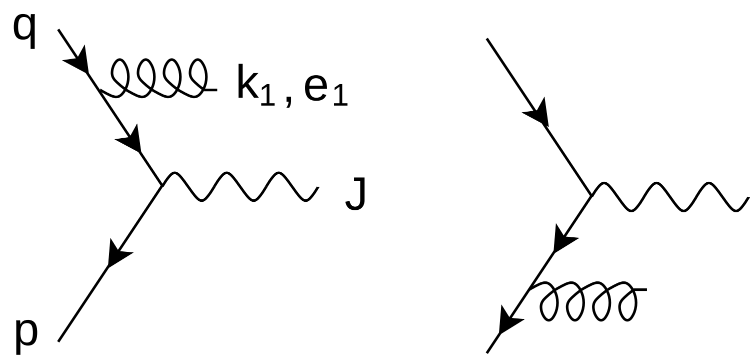
- Considering the success of the PHOTOS Monte Carlo for photon radiation, on may ask the question whether the same approach would work for QCD.
- The existence of the self-interaction of gluons seems to prohibit this.
- The importance of the study of exact QED *real*-emission matrix elements for the establishment of PHOTOS suggests one should start to study those for QCD.

- Various approaches to QCD exist which can lead to a Monte Carlo:
  - collinear limit and DGLAP-evolution;
  - $\mathrm{k_T}$-factorization and CCFM-evolution;
  - multi-Regge kinematics, BFKL.
- Can essential features of these various limits be manifestly recognized already in (exact) real-emission matrix elements?
- Can we extract approximated matrix-elements corresponding to these limits, however still valid all over phase space?

Single-emission amplitude

Cornerstone of PHOTOS is the recursive structure recognizable in QED amplitudes, the appearance of structures from lower order amplitudes in the higher order amplitudes. Therefore, we need to know the structure of single-emission amplitudes first.

We decide to study radiative corrections to the Drell-Yan like process $q\bar{q} \to J$, with the only restriction that J is color-neutral.



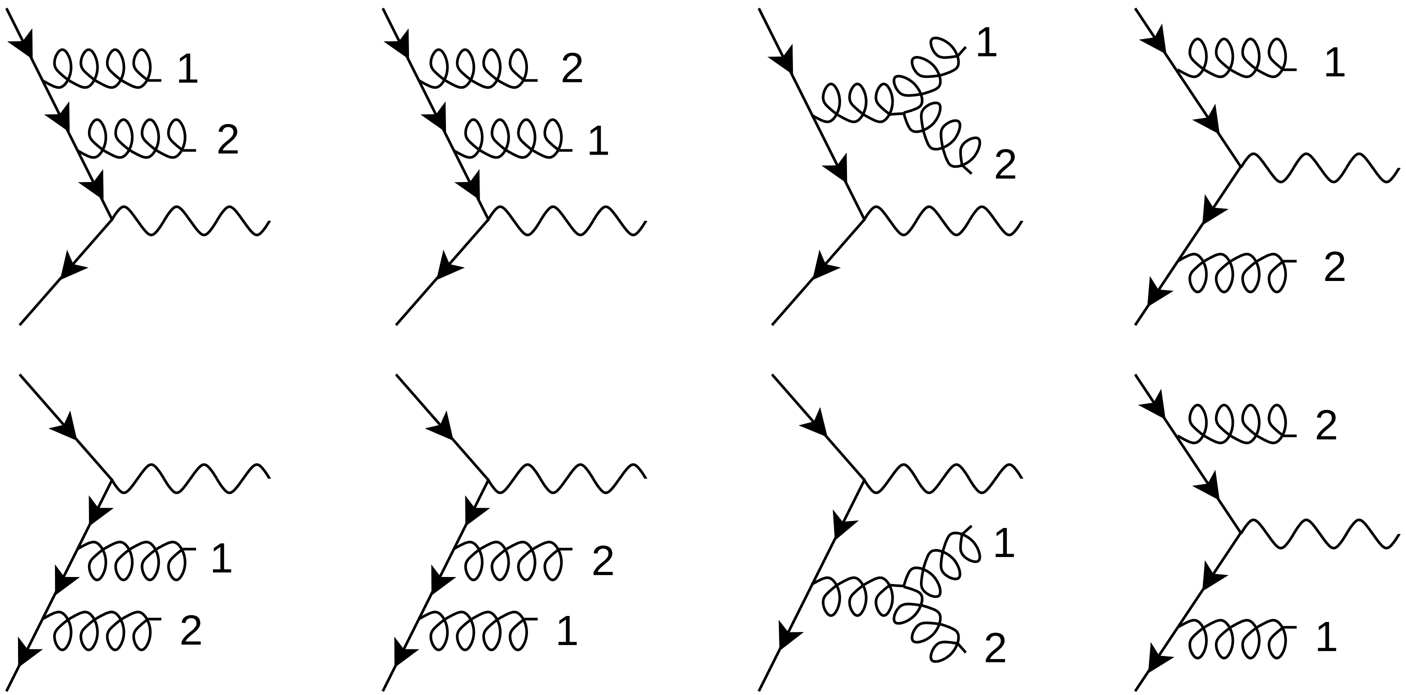
The amplitude carries one color-index associated with the gluon.

$$\mathcal{M}_1^a = \bar{v}(p) T^a I^{(1)} u(q)$$

Get rid of $\not{p}$-s and $\not{q}$-s by the Dirac-equation. Result consists of three gauge-invariant terms:

$$I^{(1)} = \not{J}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right) - \frac{1}{2}\,\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\not{J} + \frac{1}{2}\not{J}\,\frac{\not{k}_1\not{e}_1}{q\cdot k_1}$$

Double-emission amplitude



Represent amplitude in the commutator/anti-commutator color basis.

$$\mathcal{M}_2^{a,b} = \frac{1}{2}\,\overline{v}(p)\Big([T^a,T^b]I^{[1,2]} + \{T^a,T^b\}I^{\{1,2\}}\Big)u(q)$$

The QED-amplitude is proportional to $\overline{v}(p)I^{\{1,2\}}u(q)$.

Double-emission amplitude

- Apply Dirac equation

Double-emission amplitude

- Apply Dirac equation
- Avoid occurrence of $\not{J}\not{e}_{1,2}$, $\not{e}_{1,2}\not{J}$, $\not{k}_{1,2}\not{k}_{2,1}$, $\not{e}_{1,2}\not{e}_{2,1}$

Double-emission amplitude

- Apply Dirac equation
- Avoid occurrence of $\not{J}\not{e}_{1,2}$, $\not{e}_{1,2}\not{J}$, $\not{k}_{1,2}\not{k}_{2,1}$, $\not{e}_{1,2}\not{e}_{2,1}$
- Use *subtraction terms*

$$S^{\{1,2\}}_{1,q} = \frac{1}{2}\not{J}\left(\frac{q\cdot e_1}{q\cdot k_1}\,\frac{q\cdot e_2}{q\cdot k_1 + q\cdot k_2} + \frac{q\cdot e_1}{q\cdot k_1 + q\cdot k_2}\,\frac{q\cdot e_2}{q\cdot k_2}\right)$$

$$S^{\{1,2\}}_{2,q} = \frac{1}{2}\not{J}\left(\frac{\not{k}_2\not{e}_2}{q\cdot k_2}\,\frac{q\cdot e_1}{q\cdot k_1} + \frac{\not{k}_1\not{e}_1}{q\cdot k_1}\,\frac{q\cdot e_2}{q\cdot k_2}\right)$$

$$S^{\{1,2\}}_{3,q} = -\frac{1}{2}\not{J}\,\frac{k_1\cdot k_2}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\,\frac{k_1\cdot e_2\,k_2\cdot e_1}{k_1\cdot k_2\,k_1\cdot k_2}$$

and the same with momentum p. These are terms added in one place in the expression, and subtracted elsewhere in the expression.

Double-emission amplitude

- Apply Dirac equation
- Avoid occurrence of $\not{J}\not{e}_{1,2}\ ,\ \not{e}_{1,2}\not{J}\ ,\ \not{k}_{1,2}\not{k}_{2,1}\ ,\ \not{e}_{1,2}\not{e}_{2,1}$
- Use *subtraction terms*

$$S^{\{1,2\}}_{1,q} = \frac{1}{2}\not{J}\left(\frac{q\cdot e_1}{q\cdot k_1}\,\frac{q\cdot e_2}{q\cdot k_1 + q\cdot k_2} + \frac{q\cdot e_1}{q\cdot k_1 + q\cdot k_2}\,\frac{q\cdot e_2}{q\cdot k_2}\right)$$

$$S^{\{1,2\}}_{2,q} = \frac{1}{2}\not{J}\left(\frac{\not{k}_2\not{e}_2}{q\cdot k_2}\,\frac{q\cdot e_1}{q\cdot k_1} + \frac{\not{k}_1\not{e}_1}{q\cdot k_1}\,\frac{q\cdot e_2}{q\cdot k_2}\right)$$

$$S^{\{1,2\}}_{3,q} = -\frac{1}{2}\not{J}\,\frac{k_1\cdot k_2}{q\cdot k_1 + q\cdot k_2 - k_1\cdot k_2}\,\frac{k_1\cdot e_2\, k_2\cdot e_1}{k_1\cdot k_2\, k_1\cdot k_2}$$

  and the same with momentum p. These are terms added in one place in the expression, and subtracted elsewhere in the expression.
- In the following, *every line consists of a gauge-invariant expression*.

Terms contributing to the $\{T^a, T^b\}$-part

$$I_1^{\{1,2\}} = \not{J}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)$$

$$I_{2l}^{\{1,2\}} = -\frac{1}{2}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\frac{\not{e}_2\not{k}_2}{p\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}$$

$$I_{2r}^{\{1,2\}} = \frac{1}{2}\not{J}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\frac{\not{k}_2\not{e}_2}{q\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right]$$

$$I_3^{\{1,2\}} = -\frac{1}{4}\left(\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\not{J}\frac{\not{k}_2\not{e}_2}{q\cdot k_2} + \frac{\not{e}_2\not{k}_2}{p\cdot k_2}\not{J}\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right)$$

$$I_{4p}^{\{1,2\}} = \frac{1}{4}\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2}{p\cdot k_1} + \frac{\not{e}_2\not{k}_2\not{e}_1\not{k}_1}{p\cdot k_2}\right)\not{J}$$

$$I_{4q}^{\{1,2\}} = \frac{1}{4}\not{J}\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{k}_2\not{e}_2\not{k}_1\not{e}_1}{q\cdot k_1} + \frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2}{q\cdot k_2}\right)$$

Terms contributing to the $\{T^a, T^b\}$-part

$$I_{5PA}^{\{1,2\}} = \not{J}\,\frac{k_1\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{5Pb}^{\{1,2\}} = -\not{J}\,\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{k_1\cdot e_2 k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right)$$

$$I_{5QA}^{\{1,2\}} = \not{J}\,\frac{k_1\cdot k_2}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{5QB}^{\{1,2\}} = -\not{J}\,\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{k_1\cdot e_2 k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right)$$

$$I_6^{\{1,2\}} = -\frac{1}{2}\,\frac{k_1\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not{e}_2\not{k}_2}{p\cdot k_2} + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}$$

$$I_7^{\{1,2\}} = -\frac{1}{2}\,\not{J}\,\frac{k_1\cdot k_2}{q\cdot k_{1+2} - k_1\cdot k_2}\left[\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not{k}_2\not{e}_2}{q\cdot k_2} + \left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right]$$

Terms contributing to the $[T^a, T^b]$-part

$$I_1^{[1,2]} = -\frac{1}{2}\not{J}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} + \frac{q\cdot e_2}{q\cdot k_2} - 2\frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$+\frac{1}{2}\not{J}\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\left(\frac{p\cdot e_1}{p\cdot k_1} + \frac{q\cdot e_1}{q\cdot k_1} - 2\frac{k_2\cdot e_1}{k_1\cdot k_2}\right)$$

$$I_{2l}^{[1,2]} = \frac{1}{2}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\frac{\not{e}_2\not{k}_2}{p\cdot k_2} - \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}$$

$$I_{2r}^{[1,2]} = \frac{1}{2}\not{J}\left[\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right)\frac{\not{k}_2\not{e}_2}{q\cdot k_2} - \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right]$$

$$I_3^{[1,2]} = \frac{1}{4}\left(-\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\not{J}\frac{\not{k}_2\not{e}_2}{q\cdot k_2} + \frac{\not{e}_2\not{k}_2}{p\cdot k_2}\not{J}\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right)$$

$$I_{4p}^{[1,2]} = \frac{1}{4}\,\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2}{p\cdot k_1} - \frac{\not{e}_2\not{k}_2\not{e}_1\not{k}_1}{p\cdot k_2}\right)\not{J}$$

$$I_{4q}^{[1,2]} = \frac{1}{4}\not{J}\,\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2}{q\cdot k_2} - \frac{\not{k}_2\not{e}_2\not{k}_1\not{e}_1}{q\cdot k_1}\right)$$

Terms contributing to the $[T^a, T^b]$-part

$$I_{5PA}^{[1,2]} = -\not{J}\,\frac{p\cdot k_1 - p\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{5Pb}^{[1,2]} = \not{J}\,\frac{p\cdot k_1 - p\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{5QA}^{[1,2]} = -\not{J}\,\frac{q\cdot k_2 - q\cdot k_1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{5QB}^{[1,2]} = \not{J}\,\frac{q\cdot k_2 - q\cdot k_1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{6A}^{[1,2]} = \frac{1}{2}\,\frac{k_1\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2}\left[+\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not{e}_2\not{k}_2}{p\cdot k_2} - \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not{e}_1\not{k}_1}{p\cdot k_1}\right]\not{J}$$

$$I_{7A}^{[1,2]} = \frac{1}{2}\not{J}\,\frac{k_1\cdot k_2}{q\cdot k_{1+2} - k_1\cdot k_2}\left[-\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\frac{\not{k}_2\not{e}_2}{q\cdot k_2} + \left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\frac{\not{k}_1\not{e}_1}{q\cdot k_1}\right]$$

Terms contributing to the $[T^a, T^b]$-part

$$I_{6B}^{[1,2]} = \frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\left[-\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\not{e}_2\not{k}_2 + \left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\not{e}_1\not{k}_1\right]\not{J}$$

$$I_{6C}^{[1,2]} = \frac{1}{2}\,\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{e}_2\not{k}_2\not{e}_1\not{k}_1 - \not{e}_1\not{k}_1\not{e}_2\not{k}_2}{k_1\cdot k_2}\right)\not{J}$$

$$I_{7B}^{[1,2]} = \not{J}\,\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\left[+\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_1\cdot k_2}\right)\not{k}_2\not{e}_2 - \left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\not{k}_1\not{e}_1\right]$$

$$I_{7C}^{[1,2]} = -\frac{1}{2}\not{J}\,\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2 - \not{k}_2\not{e}_2\not{k}_1\not{e}_1}{k_1\cdot k_2}\right)$$

Color-ordered amplitudes

$$\mathcal{M}_2^{a,b} = \frac{1}{2}\,\overline{v}(p)\Big(T^aT^bI^{(1,2)} + T^bT^aI^{(2,1)}\Big)u(q)$$

Color-ordered amplitudes

$$\mathcal{M}_2^{a,b} = \frac{1}{2}\bar{v}(p)\Big(T^aT^bI^{(1,2)} + T^bT^aI^{(2,1)}\Big)u(q)$$

$$I^{(1,2)} = \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q\cdot k_2} + \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)$$

$$+\frac{p\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\not{J}$$

$$+\not{J}\frac{q\cdot k_1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2}\right)$$

$$+\not{J}\left(1 - \frac{p\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2} - \frac{q\cdot k_1}{q\cdot k_{1+2} - k_1\cdot k_2}\right)\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$-\frac{1}{4}\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2 - \not{e}_2\not{k}_2\not{e}_1\not{k}_1}{k_1\cdot k_2}\right)\not{J}$$

$$-\frac{1}{4}\not{J}\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2 - \not{k}_2\not{e}_2\not{k}_1\not{e}_1}{k_1\cdot k_2}\right)$$

$I^{(2,1)}$ is obtained from $I^{(1,2)}$ by interchanging k_1, e_1 with k_2, e_2.

Color-ordered amplitudes

$$\mathcal{M}_2^{a,b} = \frac{1}{2}\,\overline{v}(p)\Big(T^aT^bI^{(1,2)} + T^bT^aI^{(2,1)}\Big)u(q)$$

$$I^{(1,2)} = \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q\cdot k_2} + \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)$$

$$+\frac{p\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\not{J}$$

$$+\not{J}\,\frac{q\cdot k_1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2}\right)$$

$$+\not{J}\left(1 - \frac{p\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2} - \frac{q\cdot k_1}{q\cdot k_{1+2} - k_1\cdot k_2}\right)\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}\,\frac{k_2\cdot e_1}{k_1\cdot k_2} - \frac{e_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$-\frac{1}{4}\,\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{e}_1\not{k}_1\not{e}_2\not{k}_2 - \not{e}_2\not{k}_2\not{e}_1\not{k}_1}{k_1\cdot k_2}\right)\not{J}$$

$$-\frac{1}{4}\,\not{J}\,\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{\not{k}_1\not{e}_1\not{k}_2\not{e}_2 - \not{k}_2\not{e}_2\not{k}_1\not{e}_1}{k_1\cdot k_2}\right)$$

Notice the separate “gluon-emission factors” in the first three lines.

Mixed representation

Organize the amplitude using both color bases.

$$\mathcal{M}_2^{a,b} = \frac{1}{2}\,\overline{v}(p)\Big(T^aT^b I_{\text{mix}}^{(1,2)} + T^bT^a I_{\text{mix}}^{(2,1)} + [T^a,T^b] I_{\text{mix}}^{[1,2]} + \{T^a,T^b\} I_{\text{mix}}^{\{1,2\}}\Big)u(q)$$

The choice of coefficients is not unique; try to choose them conveniently.

Mixed representation: ordered part

Organize the amplitude using both color bases.

$$\mathcal{M}_2^{a,b} = \frac{1}{2}\,\bar{v}(p)\Big(T^aT^bI_{\text{mix}}^{(1,2)} + T^bT^aI_{\text{mix}}^{(2,1)} + [T^a,T^b]I_{\text{mix}}^{[1,2]} + \{T^a,T^b\}I_{\text{mix}}^{\{1,2\}}\Big)u(q)$$

The choice of coefficients is not unique; try to choose them conveniently.

$$\begin{aligned}
I_{\text{mix}}^{(1,2)} = {} & \left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\not{J}\left(\frac{\not{k}_2\not{e}_2}{2q\cdot k_2} + \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right) \\
& + \frac{p\cdot k_2}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{e}_1\not{k}_1}{2p\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{e}_2\not{k}_2}{2p\cdot k_2}\right)\not{J} \\
& + \not{J}\,\frac{q\cdot k_1}{q\cdot k_{1+2} - k_1\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{\not{k}_1\not{e}_1}{2q\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2} - \frac{\not{k}_2\not{e}_2}{2q\cdot k_2}\right)
\end{aligned}$$

$I_{\text{mix}}^{(2,1)}$ is chosen accordingly.

Mixed representation: $[T^a, T^b]$-part

Introduce “polarization vectors” of the virtual gluon.

$$e^{\mu}_{1+2} = \frac{k_1^{\mu} - k_2^{\mu}}{2\,k_1\cdot k_2}\left(\frac{k_1\cdot e_2\; k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right) \quad , \quad \hat{e}^{\mu}_{1+2} = \frac{k_1^{\mu} - k_2^{\mu}}{2\,k_1\cdot k_2}\;\frac{\frac{i}{4}\,\mathrm{Tr}(\gamma^5 \not{e}_1 \not{k}_1 \not{e}_2 \not{k}_2)}{k_1\cdot k_2}$$

Mixed representation: $[T^a, T^b]$-part

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$$\begin{aligned} I^{[1,2]}_{\text{mix}} &= \not{J}\left(\frac{p\cdot e_{1+2}}{p\cdot k_{1+2} - k_1\cdot k_2} - \frac{q\cdot e_{1+2}}{q\cdot k_{1+2} - k_1\cdot k_2}\right) \\ &\quad - \frac{1}{2}\,\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\,(\not{e}_{1+2} + i\gamma^5 \hat{\not{e}}_{1+2})\not{k}_{1+2}\not{J} \\ &\quad + \frac{1}{2}\not{J}\,\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\,\not{k}_{1+2}(\not{e}_{1+2} - i\gamma^5 \hat{\not{e}}_{1+2}) \end{aligned}$$

Mixed representation: $[T^a, T^b]$-part

Introduce “polarization vectors” of the virtual gluon.

$$e^\mu_{1+2} = \frac{k_1^\mu - k_2^\mu}{2\,k_1\cdot k_2}\left(\frac{k_1\cdot e_2\; k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right) \;,\quad \hat{e}^\mu_{1+2} = \frac{k_1^\mu - k_2^\mu}{2\,k_1\cdot k_2}\,\frac{\frac{i}{4}\,\mathrm{Tr}(\gamma^5 \not{e}_1 \not{k}_1 \not{e}_2 \not{k}_2)}{k_1\cdot k_2}$$

$$\begin{aligned} I^{[1,2]}_{\text{mix}} &= \not{J}\left(\frac{p\cdot e_{1+2}}{p\cdot k_{1+2} - k_1\cdot k_2} - \frac{q\cdot e_{1+2}}{q\cdot k_{1+2} - k_1\cdot k_2}\right) \\ &\quad - \frac{1}{2}\,\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\left(\not{e}_{1+2} + i\gamma^5 \hat{\not{e}}_{1+2}\right)\not{k}_{1+2}\not{J} \\ &\quad + \frac{1}{2}\,\not{J}\,\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\,\not{k}_{1+2}\left(\not{e}_{1+2} - i\gamma^5 \hat{\not{e}}_{1+2}\right) \end{aligned}$$

Notice the similarity with the single-emission amplitude.

$$I^{(1)} = \not{J}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right) - \frac{1}{2}\,\frac{\not{e}_1 \not{k}_1}{p\cdot k_1}\,\not{J} + \frac{1}{2}\,\not{J}\,\frac{\not{k}_1 \not{e}_1}{q\cdot k_1}$$

Mixed representation

Introduce “polarization vectors” of the virtual gluon.

$$e^{\mu}_{1+2} = \frac{k_1^{\mu} - k_2^{\mu}}{2\,k_1\cdot k_2}\left(\frac{k_1\cdot e_2\;k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right) \quad , \quad \hat{e}^{\mu}_{1+2} = \frac{k_1^{\mu} - k_2^{\mu}}{2\,k_1\cdot k_2}\;\frac{\frac{i}{4}\,\mathrm{Tr}(\gamma^5 \not{e}_1 \not{k}_1 \not{e}_2 \not{k}_2)}{k_1\cdot k_2}$$

$$\begin{aligned} I^{[1,2]}_{\mathrm{mix}} &= \not{J}\left(\frac{p\cdot e_{1+2}}{p\cdot k_{1+2} - k_1\cdot k_2} - \frac{q\cdot e_{1+2}}{q\cdot k_{1+2} - k_1\cdot k_2}\right) \\ &\quad - \frac{1}{2}\,\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2}\,(\not{e}_{1+2} + i\gamma^5 \hat{\not{e}}_{1+2})\not{k}_{1+2}\not{J} \\ &\quad + \frac{1}{2}\,\not{J}\,\frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\,\not{k}_{1+2}(\not{e}_{1+2} - i\gamma^5 \hat{\not{e}}_{1+2}) \end{aligned}$$

$$I^{\{1,2\}}_{\mathrm{mix}} = -\frac{1}{2}\,\not{J}\left(\frac{1}{p\cdot k_{1+2} - k_1\cdot k_2} + \frac{1}{q\cdot k_{1+2} - k_1\cdot k_2}\right)\left(\frac{k_1\cdot e_2\;k_2\cdot e_1}{k_1\cdot k_2} - e_1\cdot e_2\right)$$

- Is $I^{[1,2]}_{\mathrm{mix}}$ related to the real-emission contribution to the running coupling constant?
- Singular structures are not fully used yet to organize terms.

Ordering in the soft limit (BFKL)

Assume that $\sqrt{s} \gg k_1^0 \gg k_2^0$, and therefore $p\cdot k_1 \gg p\cdot k_2 \gg k_1\cdot k_2$ and/or $q\cdot k_1 \gg q\cdot k_2 \gg k_1\cdot k_2$. Only $I_{\text{mix}}^{(1,2)}$ and $I_{\text{mix}}^{(2,1)}$ survive in this limit.

$$I_{\text{mix}}^{(1,2)} = \not{J}\left(\frac{p\cdot e_1}{p\cdot k_1}-\frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{k_1\cdot e_2}{k_1\cdot k_2}-\frac{q\cdot e_2}{q\cdot k_2}\right)+\not{J}\left(\frac{q\cdot e_1}{q\cdot k_1}-\frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2}-\frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{\text{mix}}^{(2,1)} = \not{J}\left(\frac{p\cdot e_2}{p\cdot k_2}-\frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\left(\frac{k_2\cdot e_1}{k_2\cdot k_1}-\frac{q\cdot e_1}{q\cdot k_1}\right)+\not{J}\left(\frac{p\cdot e_2}{p\cdot k_2}-\frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\left(\frac{p\cdot e_1}{p\cdot k_1}-\frac{k_2\cdot e_1}{k_2\cdot k_1}\right)$$

All other contributions can be neglected. After some short manipulations,

$$\begin{aligned}\mathcal{M}_2^{a,b} = \frac{1}{2}\,\bar{v}(p)\not{J}\,u(q)\Bigg[&T^aT^b\left(\frac{q\cdot e_1}{q\cdot k_1}-\frac{p\cdot e_1}{p\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2}-\frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\\ &+T^bT^a\left(\frac{p\cdot e_1}{p\cdot k_1}-\frac{q\cdot e_1}{q\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2}-\frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\Bigg]\end{aligned}$$

The picture of consecutive emissions is manifest.

Soft limit without ordering

Consider the case that $\sqrt{s} \gg k_1^0, k_2^0$ and therefore $p\cdot k_1$, $p\cdot k_2 \gg k_1\cdot k_2$ and/or $q\cdot k_1$, $q\cdot k_2 \gg k_1\cdot k_2$. In this limit, also part of $I_{\text{mix}}^{[1,2]}$ survives.

$$\mathcal{M}_2^{a,b} = \frac{1}{2}\,\bar{v}(p)\Big(T^aT^b I_{\text{mix}}^{(1,2)} + T^bT^a I_{\text{mix}}^{(2,1)} + [T^a, T^b] I_{\text{mix}}^{[1,2]}\Big)u(q)$$

Soft limit without ordering

Consider the case that $\sqrt{s} \gg k_1^0, k_2^0$ and therefore $p\cdot k_1$, $p\cdot k_2 \gg k_1\cdot k_2$ and/or $q\cdot k_1$, $q\cdot k_2 \gg k_1\cdot k_2$. In this limit, also part of $I_{\text{mix}}^{[1,2]}$ survives.

$$\mathcal{M}_2^{a,b} = \frac{1}{2}\bar{v}(p)\Big(T^aT^bI_{\text{mix}}^{(1,2)} + T^bT^aI_{\text{mix}}^{(2,1)} + [T^a,T^b]I_{\text{mix}}^{[1,2]}\Big)u(q)$$

$$I_{\text{mix}}^{(1,2)} = \frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)$$
$$+ \frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{p\cdot e_1}{p\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{\text{mix}}^{[1,2]} = \frac{1}{2}\left(\frac{p\cdot k_1 - p\cdot k_2}{p\cdot k_1 + p\cdot k_2} + \frac{q\cdot k_2 - q\cdot k_1}{q\cdot k_1 + q\cdot k_2}\right)$$
$$\times\left[\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\left(\frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right) + \left(\frac{k_2\cdot e_1}{k_2\cdot k_1}\,\frac{k_1\cdot e_2}{k_2\cdot k_1} - \frac{e_2\cdot e_1}{k_2\cdot k_1}\right)\right]$$

Soft limit without ordering

Consider the case that $\sqrt{s} \gg k_1^0, k_2^0$ and therefore $p\cdot k_1,\, p\cdot k_2 \gg k_1\cdot k_2$ and/or $q\cdot k_1,\, q\cdot k_2 \gg k_1\cdot k_2$. In this limit, also part of $I_{\text{mix}}^{[1,2]}$ survives.

$$\mathcal{M}_2^{a,b} = \frac{1}{2}\,\bar{v}(p)\Big(T^aT^bI_{\text{mix}}^{(1,2)} + T^bT^aI_{\text{mix}}^{(2,1)} + [T^a,T^b]I_{\text{mix}}^{[1,2]}\Big)u(q)$$

$$I_{\text{mix}}^{(1,2)} = \frac{p\cdot k_2}{p\cdot k_1 + p\cdot k_2}\left(\frac{p\cdot e_1}{p\cdot k_1} - \frac{k_2\cdot e_1}{k_2\cdot k_1}\right)\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{q\cdot e_2}{q\cdot k_2}\right)$$
$$+ \frac{q\cdot k_1}{q\cdot k_1 + q\cdot k_2}\left(\frac{q\cdot e_1}{q\cdot k_1} - \frac{p\cdot e_1}{p\cdot k_1}\right)\left(\frac{q\cdot e_2}{q\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)$$

$$I_{\text{mix}}^{[1,2]} = \frac{1}{2}\left(\frac{p\cdot k_1 - p\cdot k_2}{p\cdot k_1 + p\cdot k_2} + \frac{q\cdot k_2 - q\cdot k_1}{q\cdot k_1 + q\cdot k_2}\right)$$
$$\times\left[\left(\frac{p\cdot e_2}{p\cdot k_2} - \frac{k_1\cdot e_2}{k_1\cdot k_2}\right)\left(\frac{k_2\cdot e_1}{k_2\cdot k_1} - \frac{q\cdot e_1}{q\cdot k_1}\right) + \left(\frac{k_2\cdot e_1}{k_2\cdot k_1}\,\frac{k_1\cdot e_2}{k_2\cdot k_1} - \frac{e_2\cdot e_1}{k_2\cdot k_1}\right)\right]$$

Terms were transported from $I_{\text{mix}}^{(1,2)}, I_{\text{mix}}^{(2,1)}$ to $I_{\text{mix}}^{[1,2]}$ compared to the original expressions.

Conclusions

- We were able to write the spin-amplitude for processes of the type $q\overline{q} \to Jgg$ in a compact form consisting of a sum of manifestly gauge-invariant terms.
- Some terms are built of gauge-invariant factors which can be assigned to separate gluons.
- Some terms can be recognized as contribution to the running of the coupling constant, and have the same shape as the single-emission amplitude.
- Left-overs carry lower singular structures.
- Soft limits lead directly to amplitudes which show the behavior of consecutive emissions, possibly with contributions to the running of the coupling constant.
- All expressions, also for the approximated amplitudes, are valid all over phase space.
- Possible crude distributions based on approximated amplitudes can easily be corrected by analytically available weights.