Solving the QCD NLO evolution equations with a Markovian Monte Carlo

WIESŁAW PŁACZEK

M. Smoluchowski Institute of Physics, Jagiellonian University, Krakow, Poland.

The project is supported by EU grant MTKD-CT-2004-510126, realized in the partnership with CERN PH/TH Division and by the Polish Ministry of Scientific Research and Information Technology grant No 620/E-77/6.PR UE/DIE 188/2005-2008

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▷ Paper: **K. Golec-Biernat, S. Jadach, WP and M. Skrzypek,**
Acta Phys. Polon. B37 (2006) 1785–1832; hep-ph/0603031.

Introduction

- **Evolution equations of the quark and gluon distributions in the hadron, known as DGLAP equations, derived in QED and QCD using the renormalization group or diagrammatic techniques can be interpreted probabilistically as a Markovian process.**
- **Such a process can be modeled using Monte Carlo methods.**
- **The corresponding MC algorithm provides, in principle, an exact solution of the evolution equations for parton distribution functions (PDFs).**
- **In practice, the main limitation of a such solution is the size of a generated MC sample, i.e. corresponding statistical errors of numerical results. This is probably the main reason why this possibility has not been exploited until recently.**
- **Instead, alternative numerical methods and programs solving the QCD evolution equations much faster than the Markovian MC have been used.**
- **Feasibility of solving efficiently the DGLAP equations at the leading-order (LO) approximation with the Markovian MC was demonstrated for the first time in: S. Jadach and M. Skrzypek, Acta Phys. Polon. B35, 745 (2004), hep-ph/0312355.**

Introduction

- The main conclusion of the above work was that the currently available computer CPU power allows to solve efficiently and precisely (at the per-mill level) the QCD evolution equations with the use of the Markovian MC algorithm.
- Of course, this method will always be slower in CPU time than non-MC techniques.
- However, it has several advantages, such as: no biases and/or numerical instabilities related to finite grids of points, use of quadratures, decomposition into finite series of polynomials, accumulation of rounding errors, etc. It is also more flexible in treatment of PDFs (e.g. no need to split them into singlet and non-singlet components) and easier to extend into higher orders, new contributions, etc.
- The above Markovian algorithm can be a basis for the final-state radiation (FSR) parton shower MC program that not only solves numerically the evolution equations but also generates events in terms of parton flavours and four-momenta.
- Moreover, this algorithm is a starting point and a testing tool for various kinds of constrained MC algorithms being developed for the initial-state radiation (ISR).
- In this talk I present the Markovian MC solution of the DGLAP evolution equations up to the next-to-leading order (NLO) in the perturbative QCD.

QCD evolution equations

▶ **The general form of the DGLAP evolution equations:**

$$\frac{\partial}{\partial \ln \mu^2} q_i = \sum_j \left(P_{q_i q_j} \otimes q_j + P_{q_i \overline{q}_j} \otimes \overline{q}_j \right) + P_{q_i G} \otimes G$$

$$\frac{\partial}{\partial \ln \mu^2} \overline{q}_i = \sum_j \left(P_{\overline{q}_i q_j} \otimes q_j + P_{\overline{q}_i \overline{q}_j} \otimes \overline{q}_j \right) + P_{\overline{q}_i G} \otimes G$$

$$\frac{\partial}{\partial \ln \mu^2} G = \sum_j \left(P_{G q_j} \otimes q_j + P_{G \overline{q}_j} \otimes \overline{q}_j \right) + P_{GG} \otimes G$$

where $\{q_1, \ldots, q_{n_f}, \overline{q}_1, \ldots, \overline{q}_{n_f}, G\}(\mu, x)$ – quark, antiquark and gluon distributions;

x – Bjorken variable; μ – hard scale, (e.g. $\mu = \sqrt{Q^2}$ in DIS).

▷ The integral convolution denoted by $\otimes$ involves only *longitudinal* momentum fractions:

$$(P \otimes q)(\mu, x) = \int_0^1 dy \int_0^1 dz\, \delta(x - zy)\, P(\alpha_s, z)\, q(\mu, y) = \int_x^1 \frac{dz}{z}\, P(\alpha_s, z)\, q\left(\mu, \frac{x}{z}\right) .$$

▷ The **splitting functions** $P(\alpha_s, z)$ depend on μ through the coupling constant $\alpha_s = \alpha_s(\mu)$:

$$P(\alpha_s, z) = \underbrace{\frac{\alpha_s}{2\pi} P^{(0)}(z)}_{\text{LO}} + \underbrace{\left(\frac{\alpha_s}{2\pi}\right)^2 P^{(1)}(z)}_{\text{NLO}} + \underbrace{\left(\frac{\alpha_s}{2\pi}\right)^3 P^{(2)}(z)}_{\text{NNLO}} + \ldots$$

QCD evolution equations

▷ From charge conjugation and $SU(n_f)$ symmetry the splitting functions P have the following general structure:

$$
\begin{aligned}
P_{q_i q_j} &= P_{\overline{q}_i \overline{q}_j} = \delta_{ij} P^V_{qq} + P^S_{qq} \\
P_{q_i \overline{q}_j} &= P_{\overline{q}_i q_j} = \delta_{ij} P^V_{q\overline{q}} + P^S_{q\overline{q}} \\
P_{q_i G} &= P_{\overline{q}_i G} = P_{FG} \\
P_{G q_i} &= P_{G \overline{q}_i} = P_{GF} \,.
\end{aligned}
$$

► **This leads to the basic form of the DGLAP evolution equations:**

$$
\begin{aligned}
\frac{\partial}{\partial \ln \mu^2} q_i &= P^V_{qq} \otimes q_i + P^V_{q\overline{q}} \otimes \overline{q}_i + P^S_{qq} \otimes \sum_j q_j + P^S_{q\overline{q}} \otimes \sum_j \overline{q}_j + P_{FG} \otimes G \\
\frac{\partial}{\partial \ln \mu^2} \overline{q}_i &= P^V_{q\overline{q}} \otimes q_i + P^V_{qq} \otimes \overline{q}_i + P^S_{q\overline{q}} \otimes \sum_j q_j + P^S_{qq} \otimes \sum_j \overline{q}_j + P_{FG} \otimes G \\
\frac{\partial}{\partial \ln \mu^2} G &= P_{GF} \otimes \sum_j (q_j + \overline{q}_j) + P_{GG} \otimes G
\end{aligned}
$$

▷ Within a given approximation some splitting functions may vanish or be equal, e.g.:

LO: $P^{V(0)}_{q\overline{q}} = P^{S(0)}_{q\overline{q}} = P^{S(0)}_{qq} = 0\,,$ **NLO:** $P^{S(1)}_{qq} = P^{S(1)}_{q\overline{q}}\,.$

▶ **Singlet case:**

$$\Sigma(\mu,x) \;=\; \sum_{j=1}^{n_f} \left[q_j(\mu,x) + \overline{q}_j(\mu,x)\right]$$

▷ Introducing the notation

$$P_{FF} = P_+^V + n_f P_+^S\,, \qquad\qquad P_+^{V,S} = P_{qq}^{V,S} \;+\; P_{q\overline{q}}^{V,S}\,,$$

we obtain the following evolution equations for the quark-singlet and gluon distributions:

$$\frac{\partial}{\partial \ln \mu^2}\,\Sigma \;=\; P_{FF} \otimes \Sigma \;+\; (2n_f P_{FG}) \otimes G$$

$$\frac{\partial}{\partial \ln \mu^2}\,G \;=\; P_{GF} \otimes \Sigma \;+\; P_{GG} \otimes G\,.$$

▷ The above splitting functions obey the general relations:

$$\int_0^1 dz\,\{\,zP_{FF}(\mu,z) + zP_{GF}(\mu,z)\,\} \;=\; \int_0^1 dz\,\{\,2n_f zP_{FG}(\mu,z) + zP_{GG}(\mu,z)\,\} \;=\; 0\,.$$

▷ This leads to the **momentum sum rule**:

$$\int_0^1 dx\,\{x\Sigma(\mu,x) \;+\; xG(\mu,x)\} \;=\; \text{const} \qquad\qquad (= 1 \quad \textbf{in parton model})$$

QCD evolution equations

▶ **Non-singlet case:**

$$V(\mu,x) \;=\; \sum_{j=1}^{n_f} \left[q_j(\mu,x) - \overline{q}_j(\mu,x)\right] \;,$$

▷ The evolution equations for the non-singlet distribution:

$$\frac{\partial}{\partial \ln \mu^2}\, V \;=\; P^V_{NS} \otimes V \;,$$

where the new splitting function:

$$P^V_{NS} = P^V_- + n_f P^S_- \;, \qquad\qquad P^{V,S}_- = P^{V,S}_{qq} - P^{V,S}_{q\overline{q}} \;.$$

▶ **The set of splitting functions (QCD kernels) usually represented in the literature:**

$$\{P^V_\pm,\, P^S_\pm,\, P_{FG},\, P_{GF},\, P_{GG}\}.$$

$P^S_+ = 0$ at **LO**, $P^S_- = 0$ at **LO** and **NLO**, others $\neq 0$ at **any order**.

▷ Having the above splitting function one can write and solve the evolution equations in any of the presented forms.

⟶ In our Monte Carlo approach we work directly in the flavour space.

Behaviour at $\mathbf{z \to 1}$

▶ The splitting functions $\{P_{\pm}^{V},\, P_{-}^{S},\, P_{GG}\}$ have the following form

$$P(\alpha_s, z) \;=\; \frac{A(\alpha_s)}{(1-z)_+} \;+\; B(\alpha_s)\,\delta(1-z) \;+\; \overline{P}(\alpha_s, z)\,,$$

▷ The functions $A(\alpha_s)$, $B(\alpha_s)$ and $\overline{P}(\alpha_s, z)$ are calculated in powers of α_s, e.g.

$$\overline{P}(\alpha_s, z) \;=\; \sum_{k=0} \alpha_s^{k+1}\, D^{(k)}(z)\,,$$

where at NLO and NNLO the coefficients $D^{(k)}(z)$ are logarithmically divergent:

$$D^{(k)}(z) \;=\; D_k\, \ln(1-z) \;+\; \mathcal{O}(1)\,.$$

▶ Similarly, the splitting functions $\{P_{FG},\, P_{GF}\}$ contain logarithmically divergent terms:

$$P(\alpha_s, z) \;=\; \begin{cases} \mathcal{O}(\alpha_s) & \text{at LO } (\mathrm{k}=0) \\ \mathcal{O}(\alpha_s^2 \ln^2(1-z)) & \text{at NLO } (\mathrm{k}=1) \\ \mathcal{O}(\alpha_s^3 \ln^4(1-z)) & \text{at NNLO } (\mathrm{k}=2). \end{cases}$$

▷ **This can lead to big positive or negative weights in Monte Carlo computations.**

Behaviour at $\mathbf{z \to 0}$

- The splitting functions $\{P_{\pm}^{V}, P_{-}^{S}\}$ are logarithmically divergent at $z = 0$ starting from NLO:

$$P(\alpha_s, z) = \sum_{k=0} \alpha_s^{k+1} \left\{ \sum_{i=1}^{2k} \overline{D}_i^{(k)} \ln^i z + \mathcal{O}(1) \right\},$$

- The remaining splitting functions $\{P_{+}^{S}, P_{FG}, P_{GF}, P_{GG}\}$ have the following behaviour:

$$P(\alpha_s, z) = E_1(\alpha_s) \frac{\ln z}{z} + E_2(\alpha_s) \frac{1}{z} + \mathcal{O}(\ln^{2k} z)\,,$$

  - ▷ The logarithmic term is present starting from the NLO $(k = 1)$ approximation:

$$E_1(\alpha_s) = \alpha_s^2 E_1^{(1)} + \alpha_s^3 E_1^{(2)} + \ldots,$$

  - ▷ while the $1/z$ term is present from the LO $(k = 0)$ approximation:

$$E_2(\alpha_s) = \alpha_s E_2^{(0)} + \alpha_s^2 E_2^{(1)} + \alpha_s^3 E_2^{(2)} \ldots.$$

QCD evolution equations

▶ The general parton–parton transition matrix for a gluon and three quark flavours $(d,\, u,\, s)$:

$$\mathbf{P}(\alpha_s, z) = \left[\begin{array}{c|ccc|ccc} P_{G\leftarrow G}, & P_{G\leftarrow d}, & P_{G\leftarrow u}, & P_{G\leftarrow s}, & P_{G\leftarrow \bar{d}}, & P_{G\leftarrow \bar{u}}, & P_{G\leftarrow \bar{s}} \\ \hline P_{d\leftarrow G}, & P_{d\leftarrow d}, & P_{d\leftarrow u}, & P_{d\leftarrow s}, & P_{d\leftarrow \bar{d}}, & P_{d\leftarrow \bar{u}}, & P_{d\leftarrow \bar{s}} \\ P_{u\leftarrow G}, & P_{u\leftarrow d}, & P_{u\leftarrow u}, & P_{u\leftarrow s}, & P_{u\leftarrow \bar{d}}, & P_{u\leftarrow \bar{u}}, & P_{u\leftarrow \bar{s}} \\ P_{s\leftarrow G}, & P_{s\leftarrow d}, & P_{s\leftarrow u}, & P_{s\leftarrow s}, & P_{s\leftarrow \bar{d}}, & P_{s\leftarrow \bar{u}}, & P_{s\leftarrow \bar{s}} \\ \hline P_{\bar{d}\leftarrow G}, & P_{\bar{d}\leftarrow d}, & P_{\bar{d}\leftarrow u}, & P_{\bar{d}\leftarrow s}, & P_{\bar{d}\leftarrow \bar{d}}, & P_{\bar{d}\leftarrow \bar{u}}, & P_{\bar{d}\leftarrow \bar{s}} \\ P_{\bar{u}\leftarrow G}, & P_{\bar{u}\leftarrow d}, & P_{\bar{u}\leftarrow u}, & P_{\bar{u}\leftarrow s}, & P_{\bar{u}\leftarrow \bar{d}}, & P_{\bar{u}\leftarrow \bar{u}}, & P_{\bar{u}\leftarrow \bar{s}} \\ P_{\bar{s}\leftarrow G}, & P_{\bar{s}\leftarrow d}, & P_{\bar{s}\leftarrow u}, & P_{\bar{s}\leftarrow s}, & P_{\bar{s}\leftarrow \bar{d}}, & P_{\bar{s}\leftarrow \bar{u}}, & P_{\bar{s}\leftarrow \bar{s}} \end{array}\right],$$

where $P_{J\leftarrow I} \equiv P_{J\leftarrow I}(\alpha_s, z)$. At the NLO, the kernels can be decomposed:

$$\mathbf{P}(\alpha_s, z) = \frac{\alpha_s(t)}{2\pi}\,\mathbf{P}^{(0)}(z) + \left(\frac{\alpha_s(t)}{2\pi}\right)^2 \mathbf{P}^{(1)}(z),$$

▷ The NLO QCD coupling in the $\overline{MS}$-scheme is

$$\alpha_s(t) = \alpha_s^{(0)}(t)\left\{1 - \alpha_s^{(0)}(t)\,\frac{b_1}{b_0}\,\ln\left(2[t - \ln\Lambda_{\overline{MS}}]\right)\right\},$$

$$b_0 = \frac{\beta_0}{4\pi}, \quad b_1 = \frac{\beta_1}{(4\pi)^2}, \quad \beta_0 = 11 - \frac{2}{3}\,n_f\,, \quad \beta_1 = 102 - \frac{38}{3}\,n_f\,, \quad t = \ln Q\,.$$

QCD evolution equations

▶ **The LO kernel matrix takes a simple form**

$$\mathbf{P}^{(0)}(z) = \left[\begin{array}{c|ccc|ccc} P^{(0)}_{GG}, & P^{(0)}_{GF}, & P^{(0)}_{GF}, & P^{(0)}_{GF}, & P^{(0)}_{GF}, & P^{(0)}_{GF}, & P^{(0)}_{GF} \\ \hline P^{(0)}_{FG}, & P^{(0)}_{FF}, & 0, & 0, & 0, & 0, & 0 \\ P^{(0)}_{FG}, & 0, & P^{(0)}_{FF}, & 0, & 0, & 0, & 0 \\ P^{(0)}_{FG}, & 0, & 0, & P^{(0)}_{FF}, & 0, & 0, & 0 \\ \hline P^{(0)}_{FG}, & 0, & 0, & 0, & P^{(0)}_{FF}, & 0, & 0 \\ P^{(0)}_{FG}, & 0, & 0, & 0, & 0, & P^{(0)}_{FF}, & 0 \\ P^{(0)}_{FG}, & 0, & 0, & 0, & 0, & 0, & P^{(0)}_{FF} \end{array}\right],$$

where

$$P^{(0)}_{GG}(z) = 2C_A \left[\frac{1}{(1-z)_+} - 2 + z(1-z) + \frac{1}{z}\right] + \frac{11C_A - 4T_f}{6}\,\delta(1-z),$$

$$P^{(0)}_{FG}(z) = T_R\,[z^2 + (1-z)^2],$$

$$P^{(0)}_{GF}(z) = C_F\,\frac{1+(1-z)^2}{z},$$

$$P^{(0)}_{FF}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\,\delta(1-z)\right],$$

The colour-group factors are: $C_A = N_c = 3,\ C_F = (N_c^2-1)/2N_c = 4/3,\ T_R = 1/2.$

QCD evolution equations

▶ **The NLO contribution to the kernel matrix is:**

$$\mathbf{P}^{(1)}(z) = \left[\begin{array}{c|ccc|ccc} P_{GG}^{(1)}, & P_{GF}^{(1)}, & P_{GF}^{(1)}, & P_{GF}^{(1)}, & P_{GF}^{(1)}, & P_{GF}^{(1)}, & P_{GF}^{(1)} \\ \hline P_{FG}^{(1)}, & P_{qq}^{V+S\,(1)}, & P_{qq}^{S\,(1)}, & P_{qq}^{S\,(1)}, & P_{q\bar{q}}^{V+S\,(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{q\bar{q}}^{S\,(1)} \\ P_{FG}^{(1)}, & P_{qq}^{S\,(1)}, & P_{qq}^{V+S\,(1)}, & P_{qq}^{S\,(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{q\bar{q}}^{V+S\,(1)}, & P_{q\bar{q}}^{S\,(1)} \\ P_{FG}^{(1)}, & P_{qq}^{S\,(1)}, & P_{qq}^{S\,(1)}, & P_{qq}^{V+S\,(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{q\bar{q}}^{V+S\,(1)} \\ \hline P_{FG}^{(1)}, & P_{q\bar{q}}^{V+S\,(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{qq}^{V+S\,(1)}, & P_{qq}^{S\,(1)}, & P_{qq}^{S\,(1)} \\ P_{FG}^{(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{q\bar{q}}^{V+S\,(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{qq}^{S\,(1)}, & P_{qq}^{V+S\,(1)}, & P_{qq}^{S\,(1)} \\ P_{FG}^{(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{q\bar{q}}^{S\,(1)}, & P_{q\bar{q}}^{V+S\,(1)}, & P_{qq}^{S\,(1)}, & P_{qq}^{S\,(1)}, & P_{qq}^{V+S\,(1)} \end{array}\right]$$

where $P_{IJ}^{(1)} \equiv P_{IJ}^{(1)}(z)$ and we use a short-hand notation: $P_{IJ}^{V+S\,(1)} \equiv P_{IJ}^{V\,(1)} + P_{IJ}^{S\,(1)}$.

▷ The non-singlet and singlet-quark kernels are given in terms of the basic NLO splitting functions P_+, P_- and P_{FF}:

$$P_{qq}^{V\,(1)} = \frac{1}{2}\left[P_+^{(1)} + P_-^{(1)}\right], \quad P_{q\bar{q}}^{V\,(1)} = \frac{1}{2}\left[P_+^{(1)} - P_-^{(1)}\right], \quad P_{qq}^{S\,(1)} = \frac{1}{2n_f}\left[P_{FF}^{(1)} - P_+^{(1)}\right].$$

▷ All the elements of the above kernel matrix are calculated from the six basic NLO splitting functions of Refs. G. Curci, W. Furmański and R. Petronzio, Nucl. Phys. **B175** (1980) 27 and W. Furmański and R. Petronzio, Phys. Lett. **B97** (1980) 437:

$$[P_+^{(1)},\, P_-^{(1)},\, P_{FF}^{(1)},\, P_{FG}^{(1)},\, P_{GF}^{(1)},\, P_{GG}^{(1)}].$$

Markovian Monte Carlo for parton-momentum distributions

- In our paper we have described a Markovian MC algorithm for **parton distributions** and we have implemented it in the MC program.
- However, the factor $1/z$ in the bremsstrahlung kernels causes a significant loss of MC efficiency!
- We can get rid of this annoying phenomenon by switching to the $xD(x)$ which evolve with the kernels $zP(z)$.
- The reason for improvement is that kernels $zP(z)$ fulfill the **momentum sum rules**.

▶ The evolution equations for $xD(x)$ read

$$\partial_t\, xD_K(t,x) = \sum_J \int_x^1 \frac{dz}{z}\; z\mathcal{P}_{KJ}(t,z)\; \frac{x}{z} D_J\left(t, \frac{x}{z}\right).$$

▷ The kernels $\mathcal{P}_{KJ}(t,z) = 2P_{KJ}(\alpha_s(t),z)$ are split into virtual and real contributions:

$$\mathcal{P}_{KJ}(t,z) = -\mathcal{P}^{\delta}_{KK}(t,\epsilon(t))\,\delta_{KJ}\,\delta(1-z) + \mathcal{P}^{\Theta}_{KJ}(t,z),$$

$$\mathcal{P}^{\Theta}_{KJ}(t,z) = \mathcal{P}_{KJ}(t,z)\,\Theta(1-z-\epsilon(t))\,\Theta(z-\epsilon'),$$

where ϵ is an infra-red (IR) cut-off.

▶ The iterative solution obtained from the above formulae reads:

$$xD_K(t,x) = e^{-\Phi_K(t,t_0)} xD_K(t_0,x) + \sum_{n=1}^{\infty} \int_0^1 dx_0 \sum_{K_0,\ldots,K_{n-1}} \prod_{i=1}^{n} \left[\int_{t_0}^{t} dt_i\, \Theta(t_i - t_{i-1}) \int_0^1 dz_i \right]$$

$$\times\, e^{-\Phi_K(t,t_n)} \prod_{i=1}^{n} \left[z_i \mathcal{P}^{\Theta}_{K_i K_{i-1}}(t_i, z_i) e^{-\Phi_{K_{i-1}}(t_i,t_{i-1})} \right] x_0 D_{K_0}(t_0,x_0)\, \delta\Big(x - x_0 \prod_{i=1}^{n} z_i\Big),$$

where $K \equiv K_n$.

▶ The running $\alpha_s(t)$ can be absorbed into the evolution variable by the transformation:

$$t \longrightarrow \tau \equiv \frac{1}{\alpha_s(t_A)} \int_{t_A}^{t} dt'\, \alpha_s(t'), \quad \frac{\partial t}{\partial \tau} = \frac{\alpha_s(t_A)}{\alpha_s(t)}.$$

▷ With the choice of $\alpha_s^{(0)}(t)$ in the definition of τ and $t_A = t_0$ we get the iterative solution:

$$xD_K(\tau,x) = e^{-\Phi_K(\tau,\tau_0)} xD_K(\tau_0,x) + \sum_{n=1}^{\infty} \int_0^1 dx_0 \sum_{K_0,\ldots,K_{n-1}} \prod_{i=1}^{n} \left[\int_{\tau_0}^{\tau} d\tau_i\, \Theta(\tau_i - \tau_{i-1}) \int_0^1 dz_i \right]$$

$$\times\, e^{-\Phi_K(\tau,\tau_n)} \prod_{i=1}^{n} \left[\mathcal{P}^{\Theta}_{K_i K_{i-1}}(\tau_i, z_i) e^{-\Phi_{K_{i-1}}(\tau_i,\tau_{i-1})} \right] x_0 D_{K_0}(\tau_0,x_0)\, \delta\Big(x - x_0 \prod_{i=1}^{n} z_i\Big),$$

where $\mathcal{P}^{\Theta}_{K_i K_{i-1}}(\tau_i, z_i) = \frac{\alpha_s^{(0)}(t_0)}{\alpha_s^{(0)}(t_i)}\, z_i \mathcal{P}^{\Theta}_{K_i K_{i-1}}(\tau_i, z_i)\,.$

Construction of MC algorithm

- We simplify the QCD kernels:

$$\mathcal{P}^{\Theta}_{IK}(\tau,z) \to \bar{\mathcal{P}}^{\Theta}_{IK}(\tau_0,z) = \Theta(1-z-\bar{\epsilon})\frac{\alpha_s^{(0)}(t_0)}{\pi} z P^{(0)}_{IK}(z),$$

$$z P^{(0)}_{IK}(z) = \frac{1}{(1-z)_+}\delta_{IK}A^{(0)}_{KK} + \delta(1-z)\delta_{IK}B^{(0)}_{KK} + F^{(0)}_{IK}(z).$$

  - ▷ The approximate kernels do not depend on τ!

- The compensating weight is:

$$\bar{w}_P = \prod_{i=1}^{n} \frac{\mathcal{P}^{\Theta}_{K_i K_{i-1}}(\tau_i, z_i)}{\bar{\mathcal{P}}^{\Theta}_{K_i K_{i-1}}(\tau_0, z_i)}.$$

- The probability of the forward Markovian leap:

$$\bar{\omega}(\tau_i, x_i, K_i | \tau_{i-1}, x_{i-1}, K_{i-1}) \equiv \Theta(\tau_i - \tau_{i-1})\ \bar{\mathcal{P}}^{\Theta}_{K_i K_{i-1}}(\tau_0, x_i/x_{i-1})\ e^{-\bar{T}_{K_{i-1}}(\tau_i,\tau_{i-1})},$$

$$\int_{\tau_{i-1}}^{\infty} d\tau_i \int_0^1 dz_i \sum_{K_i} \bar{\omega}(\tau_i, x_i, K_i | \tau_{i-1}, x_{i-1}, K_{i-1}) \equiv 1\,, \quad z_i = x_i/x_{i-1}.$$

▶ The real-emission form factor is defined as follows

$$\bar{T}_K(\tau_i,\tau_{i-1}) = \int_{\tau_{i-1}}^{\tau_i} d\tau' \int_0^1 dz \sum_J \bar{\mathcal{P}}^{\Theta}_{JK}(\tau_0,z)$$

$$= (\tau_i-\tau_{i-1})\frac{\alpha_s^{(0)}(t_0)}{\pi}\left[A^{(0)}_{KK}\ln\frac{1}{\bar{\epsilon}} + \sum_J \int_0^1 F^{(0)}_{JK}(z)dz\right]$$

$$= (\tau_i-\tau_{i-1})\sum_J \bar{\pi}_{JK} = (\tau_i-\tau_{i-1})\,\bar{R}_K.$$

▶ On the other hand, the exact virtual (Sudakov) form factor is

$$\Phi_K(\tau,\tau_0) = \int_{\tau_0}^{\tau} d\tau' \,\frac{\alpha_s^{(0)}(t_0)}{\alpha_s^{(0)}(t')}\,2\left[A_{KK}(\tau')\ln\frac{1}{\epsilon(\tau')} - B_{KK}(\tau')\right].$$

▷ At LO, for the one-loop $\alpha_s^{(0)}$ and $\epsilon(\tau)=\epsilon=const$, it becomes simply

$$\Phi_K(\tau,\tau_0) = (\tau-\tau_0)\frac{\alpha_s^{(0)}(t_0)}{\pi}\left(A^{(0)}_{KK}\ln\frac{1}{\epsilon} - B^{(0)}_{KK}\right).$$

▷ At NLO it is much more complicated, but can be integrated analytically (see our paper).

- To complete the Markovianization, the integral over the "spill-over" variable τ_{n+1} is added with the help of the identity

$$e^{-\Phi_{K_n}(\tau,\tau_n)} = e^{\bar{\Delta}_{K_n}(\tau,\tau_n)} \int_\tau^\infty d\tau_{n+1} \int_0^1 dz_{n+1} \sum_{K_{n+1}} \bar{\omega}(\tau_{n+1}, x_{n+1}, K_{n+1}|\tau_n, x_n, K_n)\,,$$

where $z_{n+1} = x_{n+1}/x_n$ and

$$\bar{\Delta}_K(\tau_i,\tau_{i-1}) = \bar{T}_K(\tau_i,\tau_{i-1}) - \Phi_K(\tau_i,\tau_{i-1}) = (\tau_i - \tau_{i-1})\bar{R}_K - \Phi_K(\tau_i,\tau_{i-1}).$$

- The advantage this method is that at the LO for $\epsilon = \bar{\epsilon}$ we obtain

$$\bar{\Delta}_K = 0$$

due to the fact that the kernels obey the **momentum sum rules**.

$\longrightarrow$ This is also valid at the NLO in the $\overline{MS}$ scheme.

$\triangleright$ In actual MC calculation $\bar{\Delta}_K$ can be non-zero due to simplifications in the QCD kernels at the low MC generation level.

▶ **The final formula for this MC scenario with the importance sampling for the running α_s:**

$$xD_K(\tau,x) = e^{\bar{\Delta}_K(\tau,\tau_0)} \int\limits_{\tau_1>\tau} d\tau_1 dz_1 \sum_{K_1} \bar{\omega}(\tau_1, z_1 x, K_1|\tau_0, x, K)\ \ xD_K(\tau_0,x)$$

$$+\sum_{n=1}^{\infty} \int\limits_0^1 dx_0 \int\limits_{\tau_{n+1}>\tau} d\tau_{n+1} dz_{n+1} \sum_{K_{n+1}} \sum_{K_0,\ldots,K_{n-1}} \prod_{i=1}^{n} \int\limits_{\tau_i<\tau}^{t} d\tau_i dz_i$$

$$\times\, \bar{\omega}(\tau_{n+1}, x_{n+1}, K_{n+1}|\tau_n, x_n, K_n) \prod_{i=1}^{n} \bar{\omega}(\tau_i, x_i, K_i|\tau_{i-1}, x_{i-1}, K_{i-1})$$

$$\times\, \delta\Big(x - x_0 \prod_{i=1}^{n} z_i\Big)\ x_0 D_{K_0}(\tau_0, x_0)\ \bar{w}_P\, \bar{w}_\Delta .$$

where:

$$z_i = \frac{x_i}{x_{i-1}}, \qquad K \equiv K_n,$$

and

$$\bar{w}_\Delta = e^{\bar{\Delta}_{K_n}(\tau,\tau_n)} \prod_{i=1}^{n} e^{\bar{\Delta}_{K_{i-1}}(\tau_i,\tau_{i-1})} .$$

Numerical tests

- We have implemented the above Markovian MC algorithm up to NLO in the MC program **EvolFMC**.
- Then, we have performed comparisons of the MC solution of the DGLAP with another solution provided by the non-MC program **QCDnum16** (M. Botje, ZEUS Note 97-066, http://www.nikhef.nl/ h24/qcdcode/).
- In both cases we have evolved singlet PDF for gluons and three doublets of massless quarks from $Q_0 = 1$ GeV to $Q = 10, 100, 1000$ GeV.

▶ In our test, we have used the following parameterization of the starting parton distributions in the proton at $Q_0 = 1$ GeV:

$$xD_G(x) = 1.9083594473 \cdot x^{-0.2}(1-x)^{5.0},$$
$$xD_q(x) = 0.5 \cdot xD_{\rm sea}(x) + xD_{2u}(x),$$
$$xD_{\bar{q}}(x) = 0.5 \cdot xD_{\rm sea}(x) + xD_d(x),$$
$$xD_{\rm sea}(x) = 0.6733449216 \cdot x^{-0.2}(1-x)^{7.0},$$
$$xD_{2u}(x) = 2.1875000000 \cdot x^{0.5}(1-x)^{3.0},$$
$$xD_d(x) = 1.2304687500 \cdot x^{0.5}(1-x)^{4.0},$$

Numerical tests

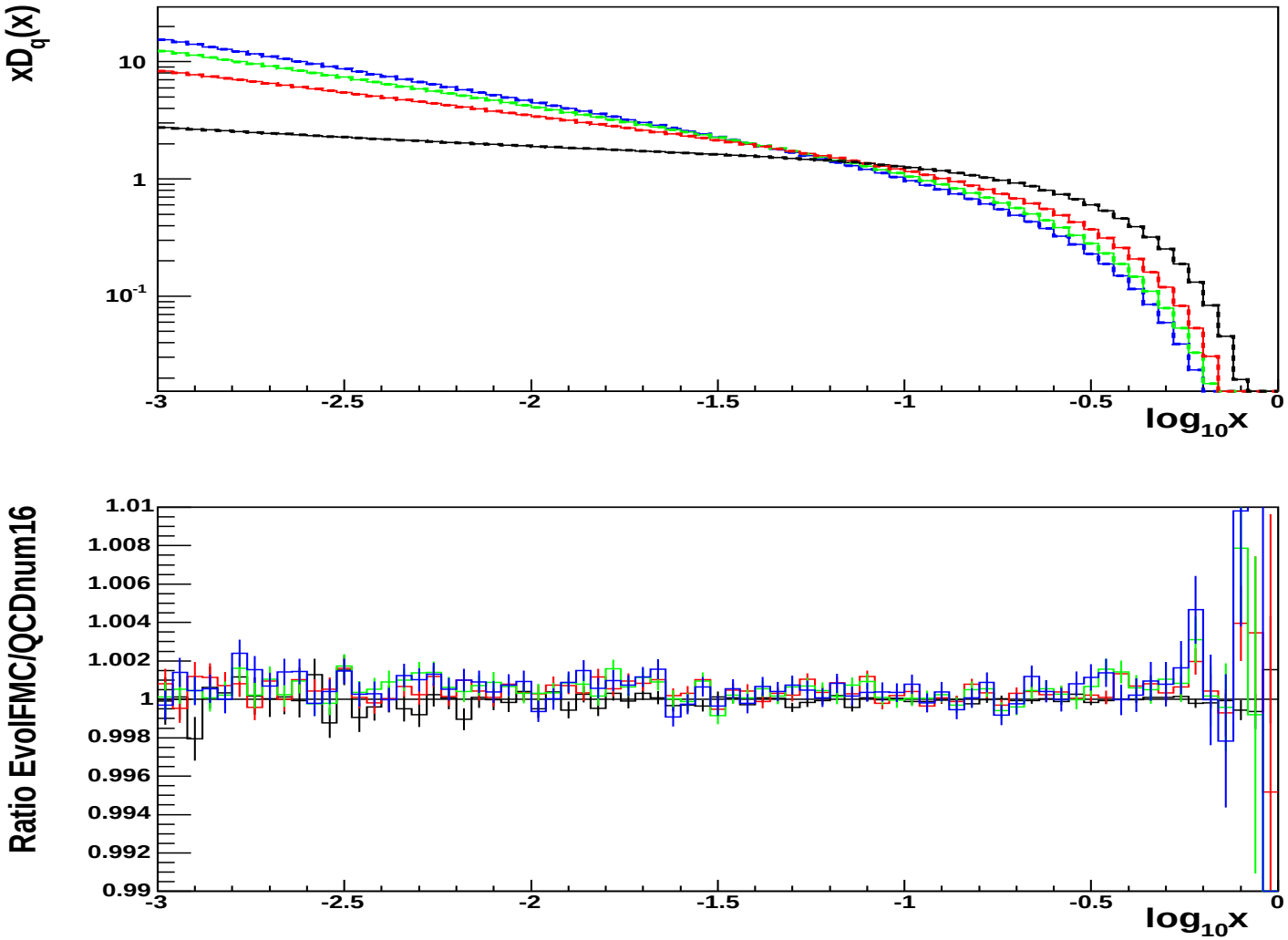

Figure 1: The upper plot shows the **quark** distribution $xD_q(x, Q_i)$ evolved from $Q_0 = 1$ GeV (black) to $Q_i = 10$ (red), 100 (green) and 1000 (blue) GeV, obtained in the NLO approximation from **EvolFMC** (solid lines) and **QCDnum16** (dashed lines), while the lower plot shows their ratio.

Numerical tests

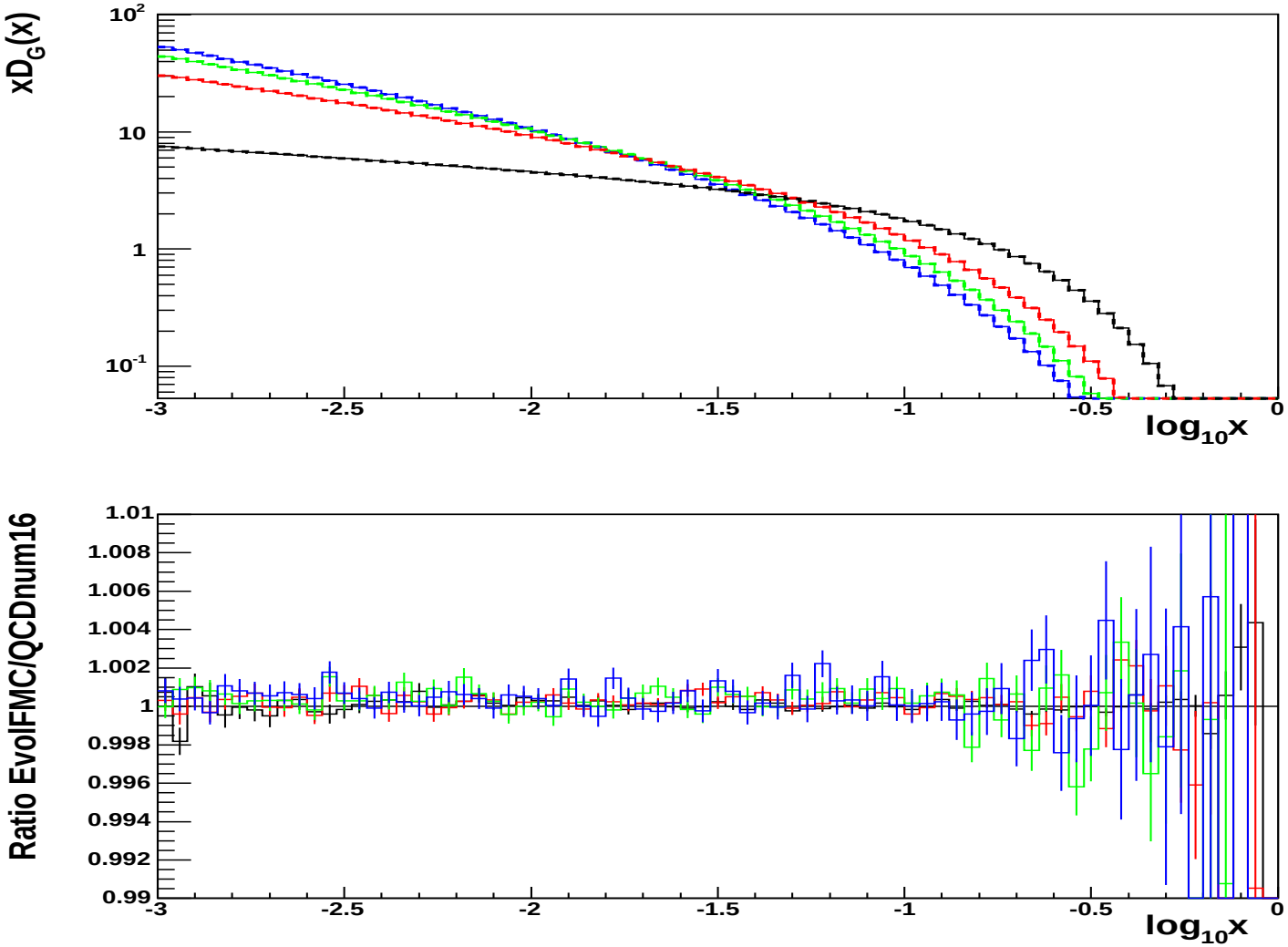

Figure 2: The upper plot shows the **gluon** distribution $xD_G(x, Q_i)$ evolved from $Q_0 = 1$ GeV (black) to $Q_i = 10$ (red), 100 (green) and 1000 (blue) GeV, obtained in the NLO approximation from **EvolFMC** (solid lines) and **QCDnum16** (dashed lines), while the lower plot shows their ratio.

Summary and outlook

- We have constructed the **Markovian Monte Carlo algorithm** for solving the **QCD DGLAP evolution equations** at the **NLO**.
- We have implemented this algorithm in the **MC program EvolFMC** (in C++).
- We have cross-checked **EvolFMC** with the non-MC programs **QCDnum16** and **APCheb33** (of K. Golec-Biernat) and found **agreement** at the **per-mill level**.
- MC computation for the **NLO** evolution is $\sim \mathbf{5}$ **times slower** than for the **LO** evolution.
- Singular behaviour of the **NLO** P_{FG} and P_{GF} **splitting functions** at $z \to 1$ leads to **large positive weights** for the $F \to G$ transitions and to **negative weights** for the $G \to F$ transitions in the region of $z \gtrsim 0.95$. $\to$ **Resummation needed!**
- So far only **massless quarks** have been considered, however, adding **heavy quarks** can be done rather easily.
- Also extension to the **NNLO** seems to be straightforward.
- This program can be used as a **testing tool** for **constrained MC algorithms for the ISR**.
- This algorithm can be a basis for the **FSR parton shower MC event generator**.