Epiphany Conference, January, 4-6 2007, Cracow

Infrared Evolution Equations method and its applications

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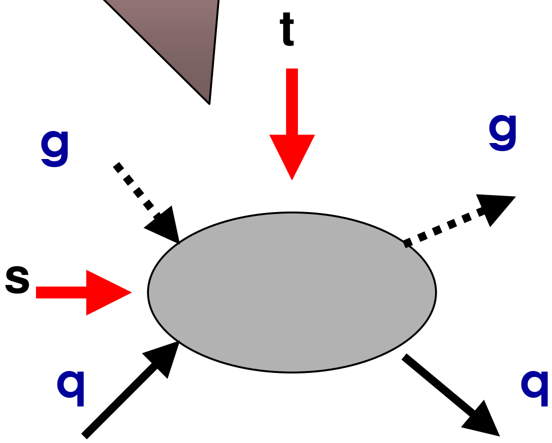
Highlights of the history of the method

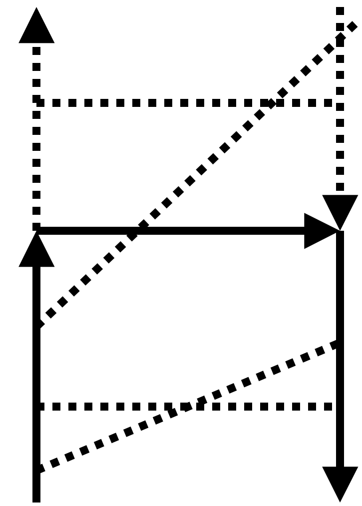
- Analyses of two-particle cuts in Regge kinematics — Gribov
- Factorization of photons with small transverse momenta — Gribov
- Infrared cut-off in the transverse momentum space — Lipatov
- Quark-quark scattering amplitudes — Kirschner-Lipatov
- Generalization of Gribov bremsstrahlung theorem to QCD , inelastic quark form factors — Ermolaev-Fadin -Lipatov
- QCD inelastic processes in Regge kinematics — Ermolaev-Lipatov
- Applications to Polarized Deep-Inelastic scattering — Bartels-Ermolaev -Manaenkov-Ryskin- Greco-Troyan
- Applications to electroweak reactions — Fadin-Lipatov-Martin-Melles, Ermolaev- Greco-Troyan

Scattering amplitude for a QED, QCD, or EW reaction



Essence of the method

Typical Feynman graph



In order to regulate IR divergences ntroduce IR cut-off μ for all virtual particles:

$$k_{\perp} > \mu$$

Lipatov

DL contributions arrive from the integration region where $k_{\perp} < k_{\parallel}$ →

the cut-off works both in the longitudinal and in the transverse space

DL contributions come from the region where transverse momenta are widely different → one can factorize the phase space into a set of separable sub-regions, in each region some virtual particle has minimal $k_\perp$

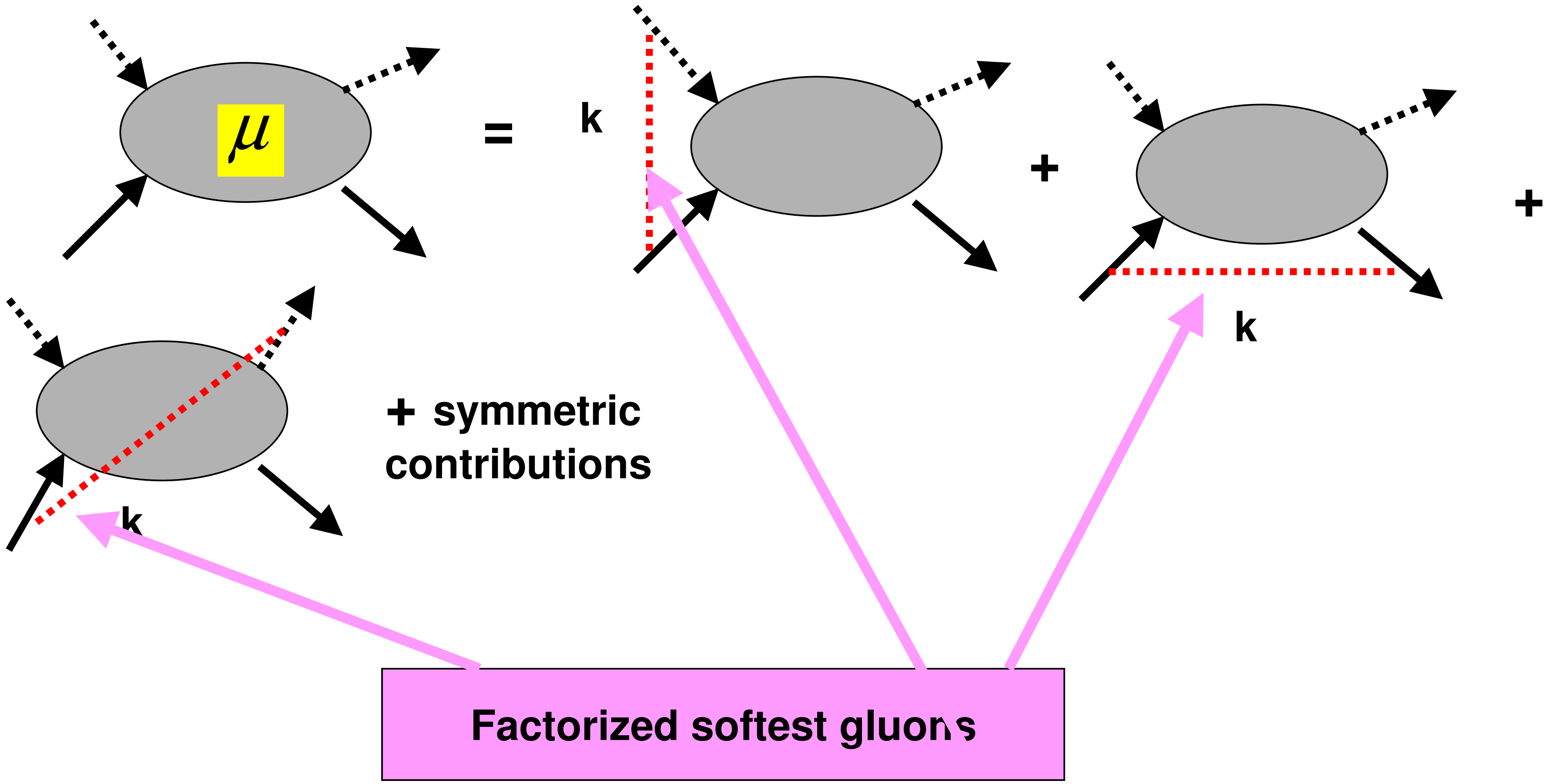
Let us call such a particle the softest one.

DL contributions of softest particles can be factorized and their transverse momentum plays the role of a new IR cut-off for integration over other virtual momenta

When μ >> involved masses, one can drop the masses and be free of IR singularities.

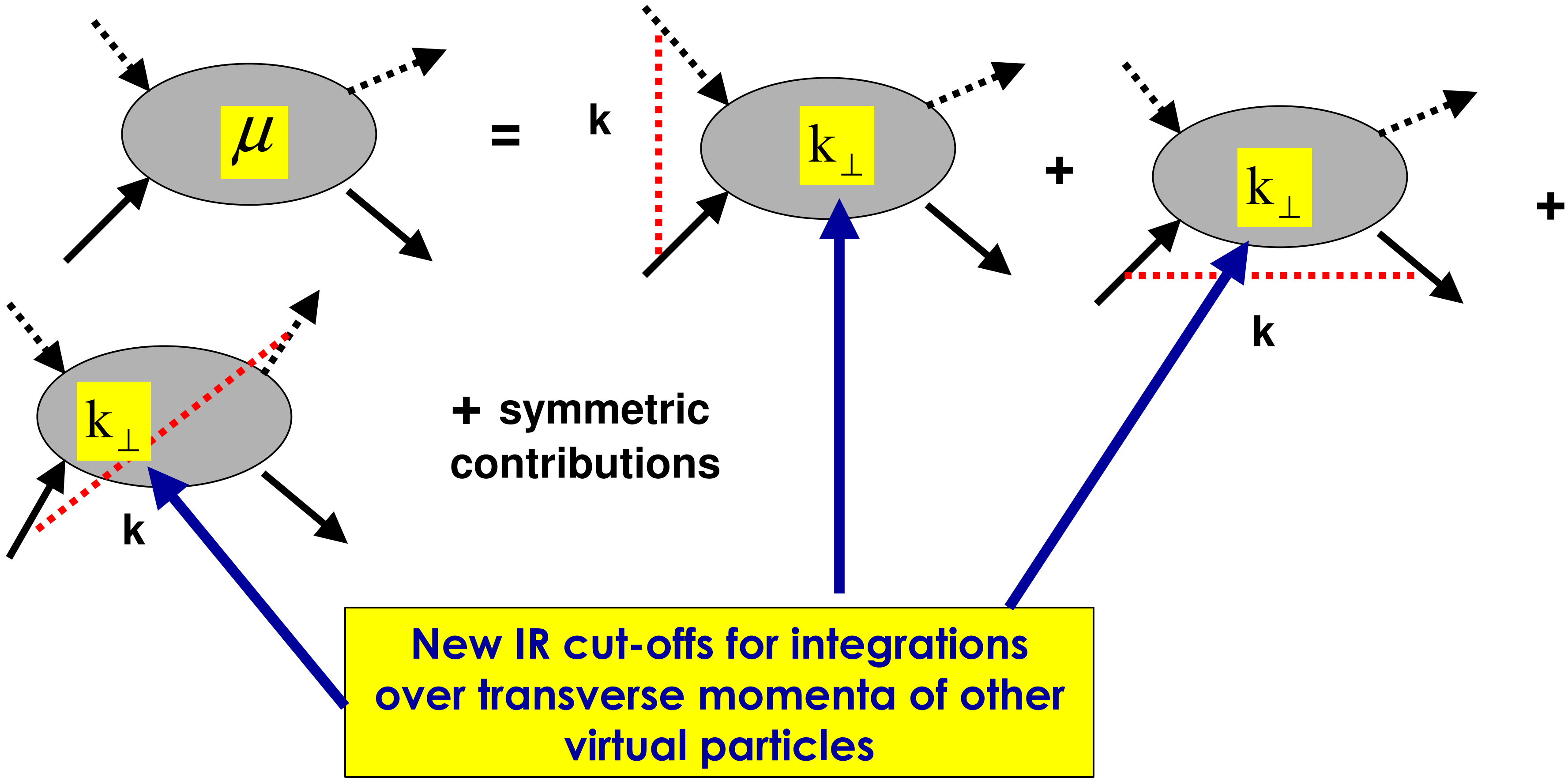
The softest particle can be either boson (gluon) or fermion (quark)

Case A: the softest particle is gluon. It can be factorized in this way:



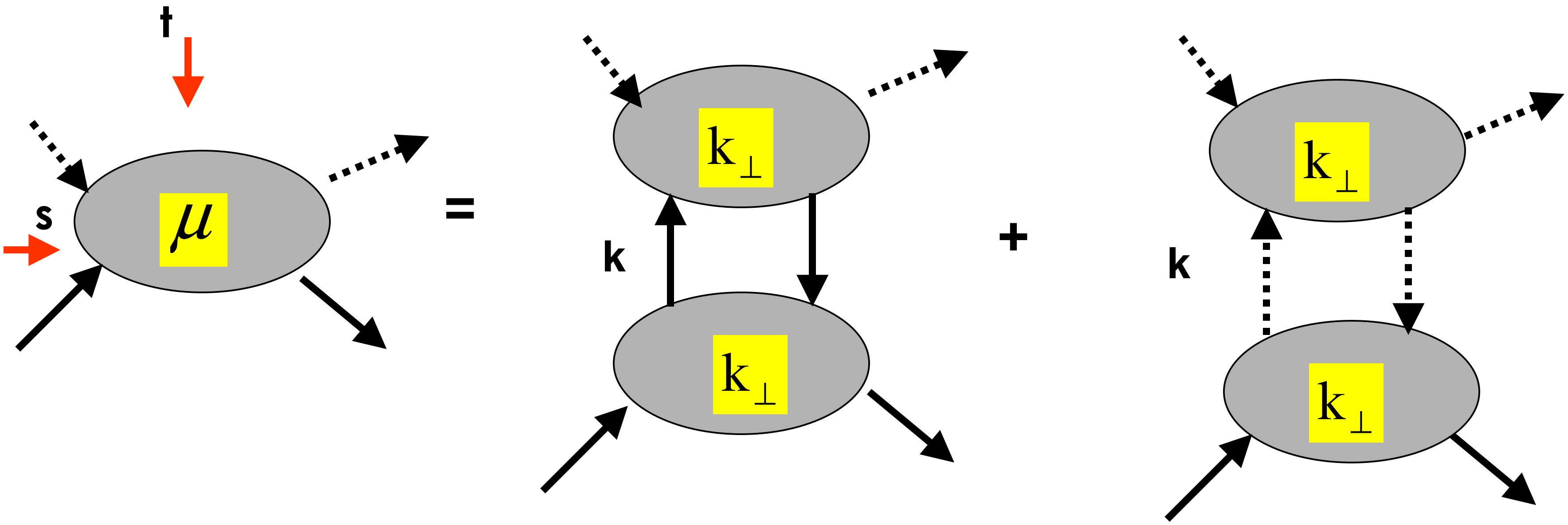
μ is the lowest limit for integration over $k_\perp$ of the softest gluon only.
It does not involve other momenta

$$\int_{\mu} dk_\perp F(k_\perp, s, t)$$



μ Is replaced by $k_\perp$ In the blobs with factorized gluons

Case B. The softest particle is a quark. It can also be factorized:



DL contributions arrive from the region $-t < k_\perp^2 < s$ →

Case B is absent when s ~ - t ⟷ hard kinematics but contributes to Regge kiinematics

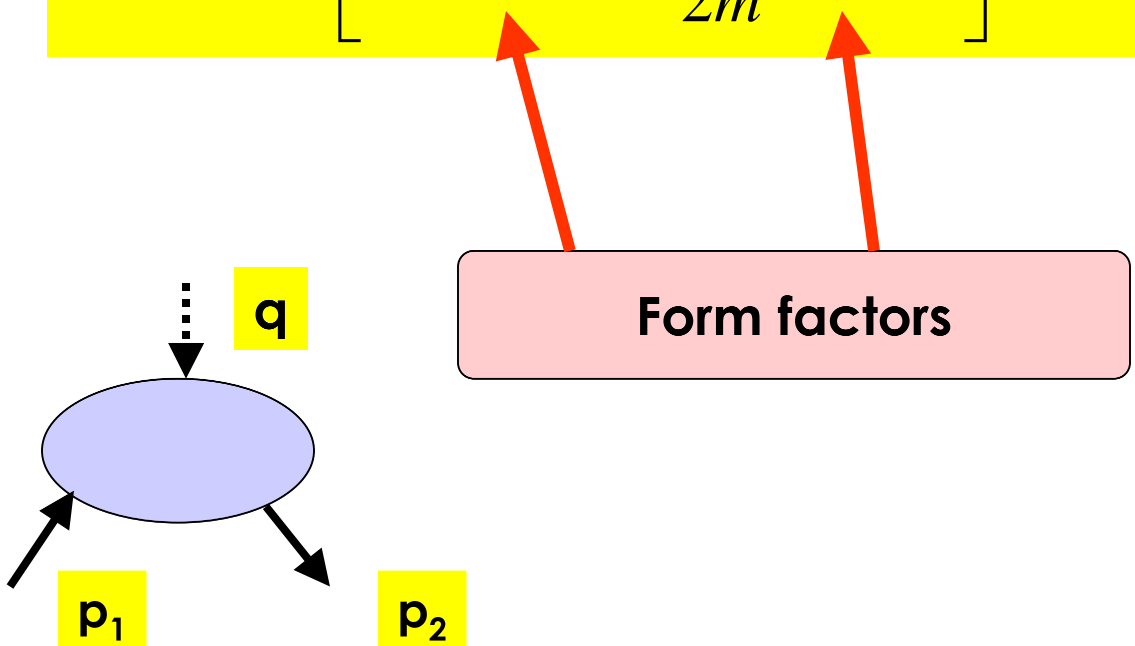
Combining cases A and B and adding Born contributions leads to IREE

Simplest application: Asymptotics of form factors of electron and quark

Electromagnetic fermion vertex:

$$\Gamma_\mu = \overline{u(p_2)}\left[\gamma_\mu f(q^2) - \frac{\sigma_{\mu\nu} q_\nu}{2m} g(q^2)\right] u(p_1)$$

q = $p_2 - p_1$



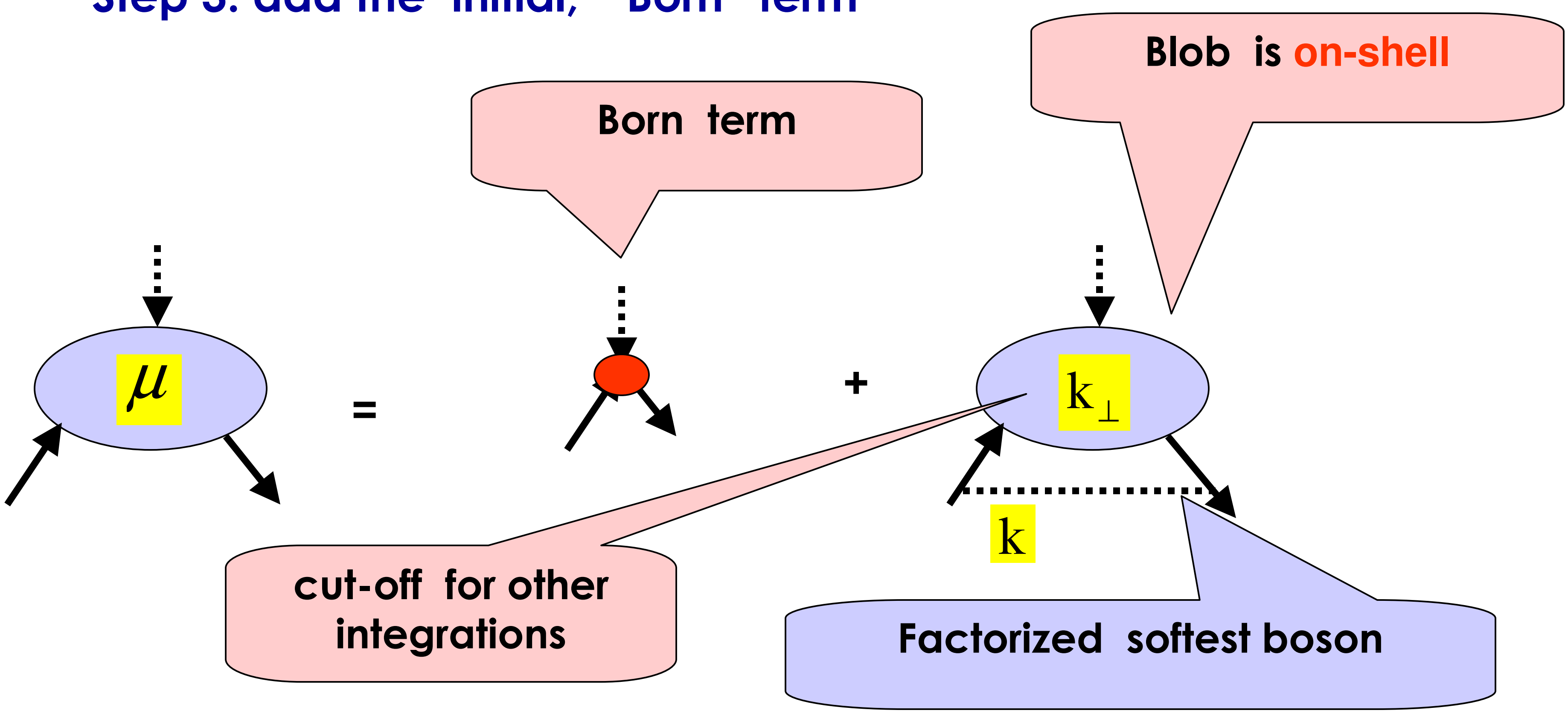
Consider the vertex in the kinematics: $|q^2| \gg p_1^2 = p_2^2 = m^2$

Composing IREE for the form factors.

Step 1: introduce μ ,

Step 2: factorize the softest photon or gluon,

Step 3: add the initial, “Born” term



IREE for form factor f in the integral form

$$f(q^2/\mu^2) = f_{Born} - \frac{a}{2\pi}\int_{\mu^2}^{q^2}\frac{dk_\perp^2}{k_\perp^2}\ln\left(\frac{q^2}{k_\perp^2}\right)f(q^2/k_\perp^2)$$

Result of integrating over longitudinal k

solution

$$f = f_{Born}\exp\left(-\frac{a}{4\pi}\ln^2(q^2/\mu^2)\right)$$

For form factor $f(q^2)$, the Born term = 1

For form factor $g(q^2)$, the "Born " term = $-\frac{m^2}{q^2}\frac{a}{\pi}\ln\left(\frac{q^2}{m^2}\right)$,

where

$$a_{QED}=\alpha, \quad a_{QCD}=\alpha_s C_F$$

Solution

$$\Gamma_\mu=\left[\gamma_\mu+\frac{\sigma_{\mu\nu}q_\nu}{m}\frac{\partial}{\partial(q^2/m^2)}\right]\exp\left(-\frac{a}{4\pi}\frac{q^2}{m^2}\right)$$

Ermolaev- Troyan

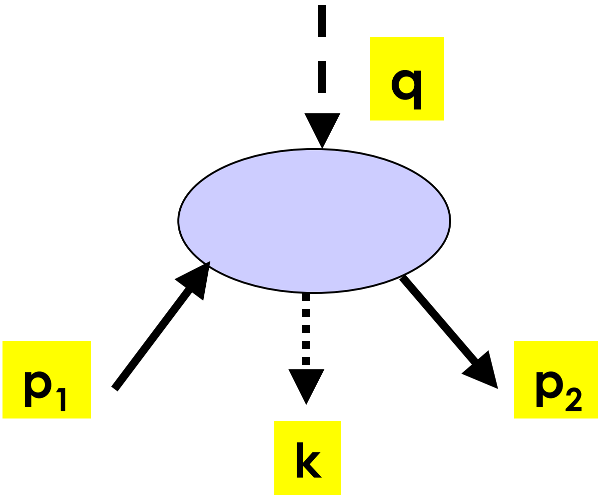
Sudakov

where is assumed

$$\mu=m$$

Mass of the fermion

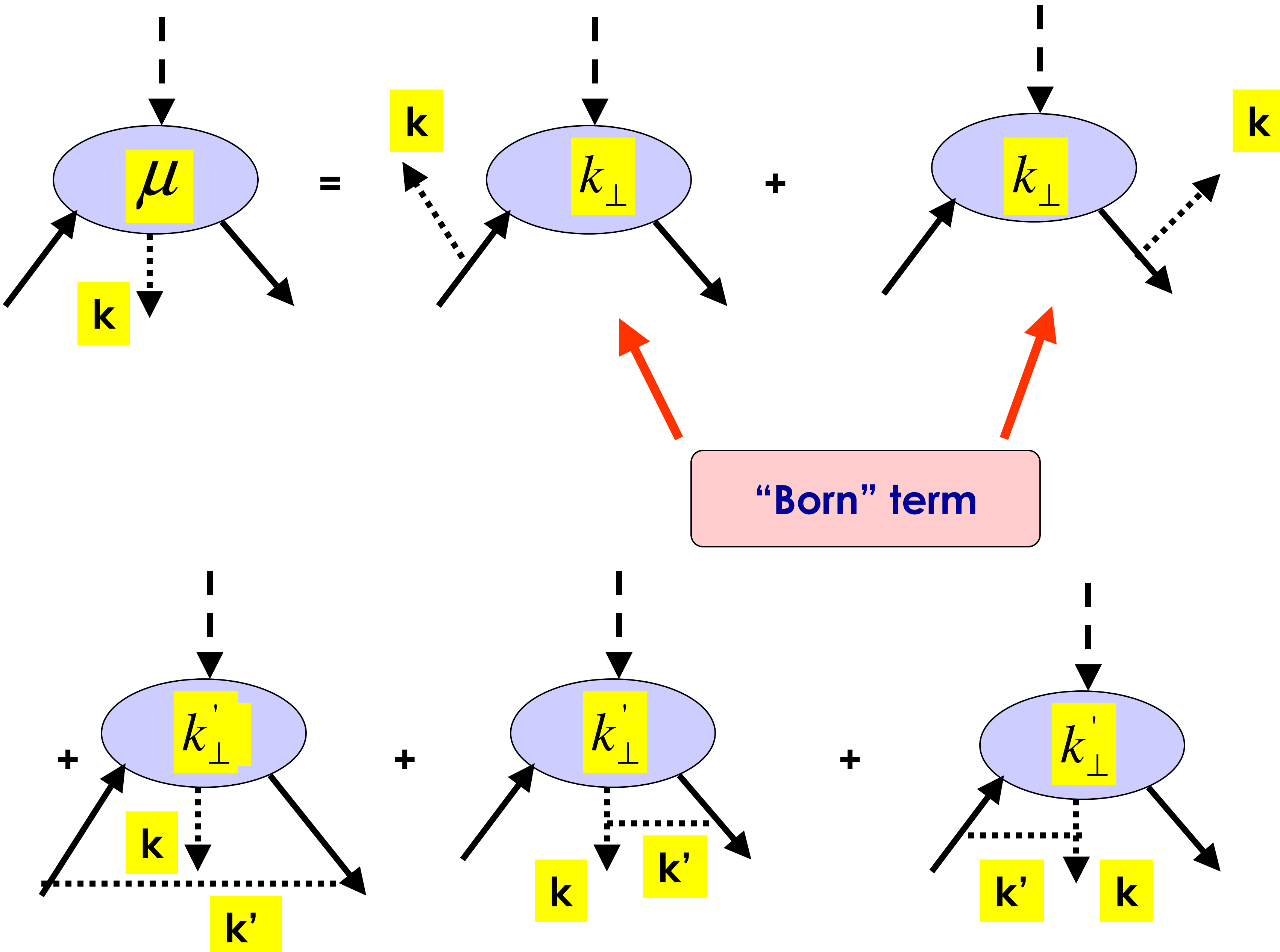
Inelastic form factor of a quark $e^+e^- \to q\bar{q} + \mathrm{n}\ gluons$



IREE for the form factor: look for the softest boson

Step 1: factorize the emitted gluon in the region $k_\perp << k_\perp^{'}$

Step 2: factorize the softest virtual gluon in the region $k_\perp >> k_\perp^{'}$



μ
k
=
k
$k_\perp$
+
$k_\perp$
k
"Born" term
+
$k'_\perp$
k
k'
+
$k'_\perp$
k
k'
+
$k'_\perp$
k'
k

Solution

$$F = \exp\left(-\frac{\alpha_s}{4\pi}\left[C_F \ln^2(q^2/m^2) + (N/2)\ln^2(k_\perp^2/m^2)\right]\right)$$

Ermolaev-
Fadin-
Lipatov

$$G = -2\frac{\partial F}{\partial(q^2/m^2)}$$

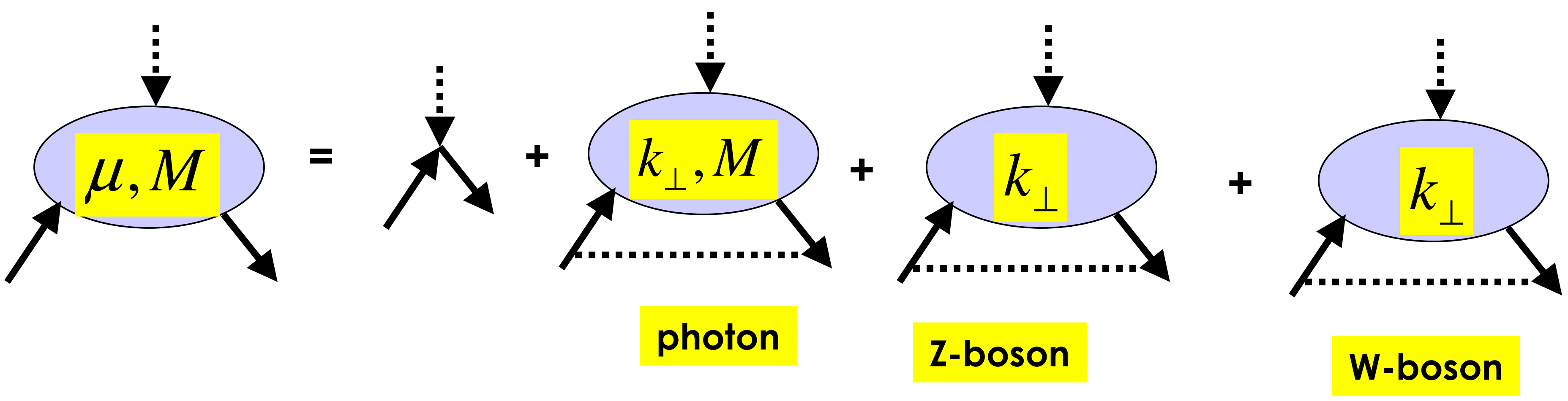
Ermolaev- Troyan

Expressions for emission of n gluons are obtained similarly

Exponentiation of electroweak double logarithms

Mass scale is more involved: μ , M_Z, M_W

Assumption: $M_Z = M_W = M$ and M can be treated as the second cut-off



Solution

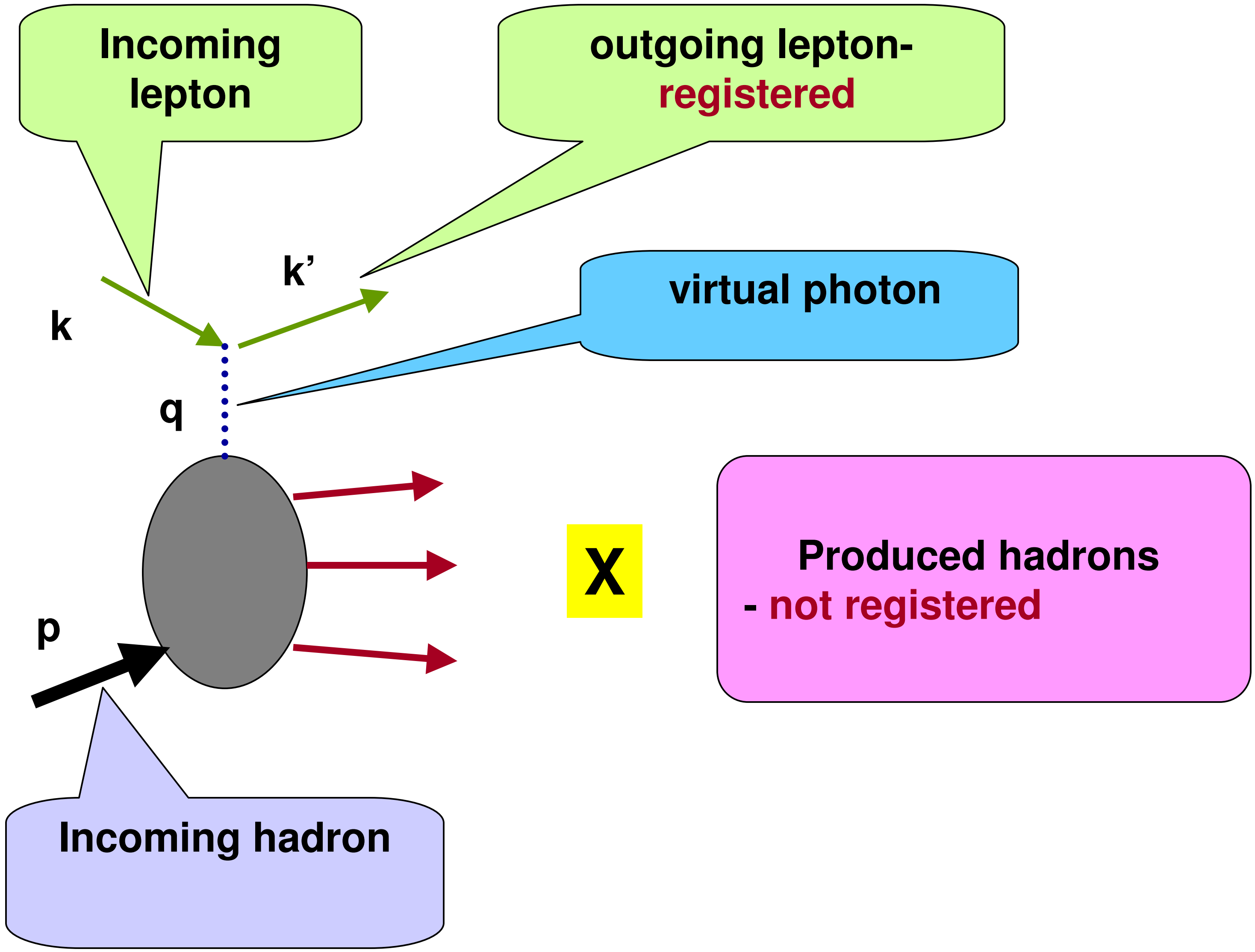
Fadin-Lipatov-Martin-Melles

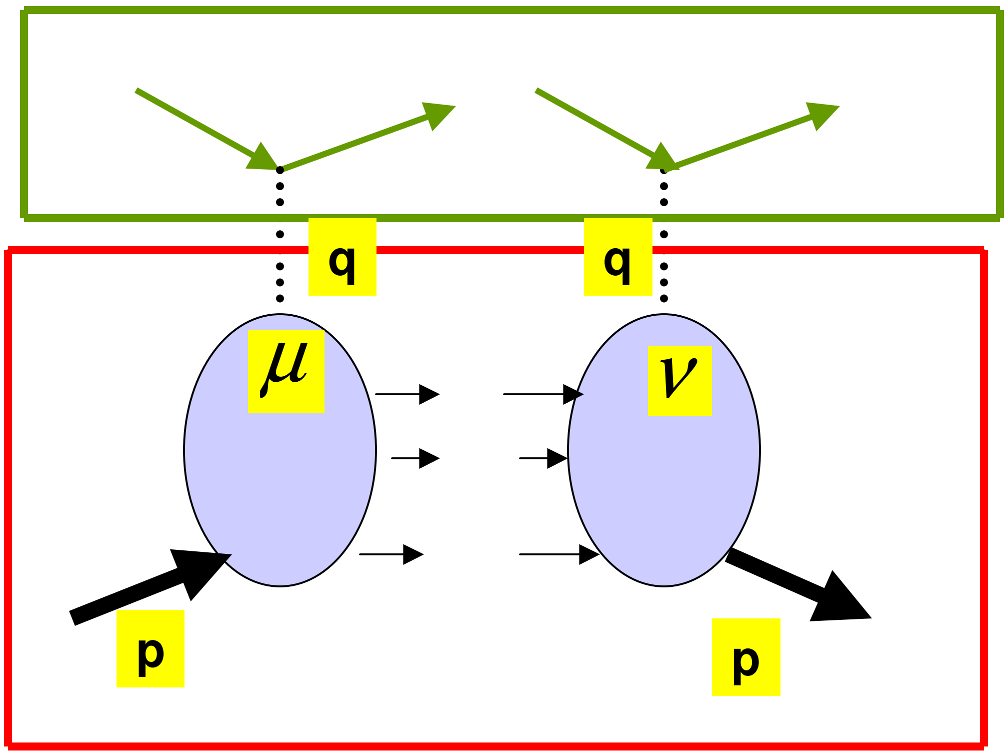
$$F = \exp\left(-\frac{\alpha Q^2}{4\pi} \ln^2(q^2/\mu^2) - \left[\frac{g^2 C_F}{16\pi^2} + \frac{g'^2}{16\pi^2}\left((Y_1^2 + Y_2^2)/4\right) - \frac{\alpha Q^2}{4\pi} \right] \ln^2(q^2/M^2) \right)$$

SU(2) - coupling

U(1) - coupling

Deep Inelastic Scattering





Leptonic tensor

hadronic tensor $W_{\mu\nu}$

Does not depend on spin

Spin-dependent

$$W_{\mu\nu} = W_{\mu\nu}^{unpolarized}(p,q) + W_{\mu\nu}^{spin}(p,q)$$

symmetric

antisymmetric

The spin-independent part of $W_{\mu\nu}$ is parameterized by two structure functions:

$$W_{\mu\nu}^{unpol} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_1(x,Q^2) + \left(p_\mu - q_\mu \frac{pq}{q^2} \right)\left(p_\nu - q_\nu \frac{pq}{q^2} \right) \frac{F_2(x,Q^2)}{pq}$$

Projection operators respect Lorentz and gauge symmetries

$$q_\mu W_{\mu\nu} = q_\nu W_{\mu\nu} = 0$$

where p is the hadron momentum, q is the virtual photon momentum ($Q^2 = -q^2 > 0$). Both of the functions depend on Q^2 and $x = Q^2/2pq$, $0 < x < 1$.

$F_2 \longrightarrow 2xF_1$ when $x \longrightarrow 0$

In the QCD framework, he spin-dependent part of $W_{\mu\nu}$ is also parameterized by two structure functions:

$$W^{spin}_{\mu\nu} = \frac{m}{pq} i\varepsilon_{\mu\nu\lambda\rho} q_\lambda \left[S_\rho g_1(x,Q^2) + \left(S_\rho - \frac{Sq}{pq} p_\rho \right) g_2(x,Q^2) \right]$$

where m, p and S are the hadron mass, momentum and spin; q is the virtual photon momentum ($Q^2 = -q^2 > 0$). Again both functions depend on Q^2 and $x = Q^2/2pq$, $0 < x < 1$. They measure asymmetries

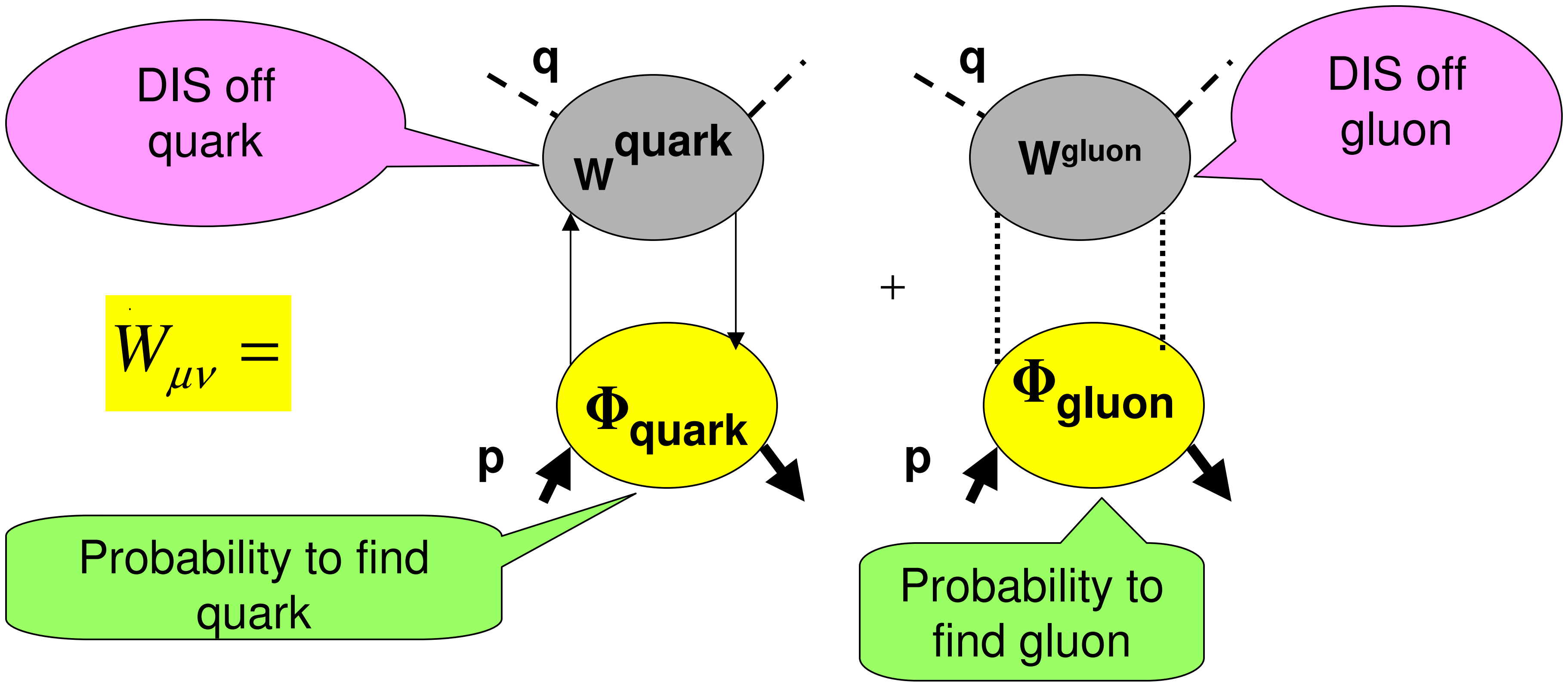
g_1 measures the longitudinal spin flip

$$g_1 \propto \sigma_{L\uparrow\uparrow} - \sigma_{L\uparrow\downarrow}$$

$g_1 + g_2$ measures the transverse spin flip

$$g_1 + g_2 \propto \sigma_{T\uparrow\uparrow} - \sigma_{T\uparrow\downarrow}$$

FACTORISATON: $W_{\mu\nu}$ **is a convolution of the the partonic tensor and probabilities to find a polarized parton (quark or gluon) in the hadron :**



DIS off quark and gluon can be studied with perturbative QCD, with calculating involved Feynman graphs.

Probabilities, Φ_{quark} and Φ_{gluon} involve non-perturbaive QCD. There is no a regular analytic way to calculate them. Usually they are defined from experimental data at large x and small Q^2 , they are called the initial quark and gluon densities and are denoted δq and δg .

So, the conventional form of the hadronic tensor is:

$$W_{\mu\nu} = W_{\mu\nu}^{quark} \otimes \delta q + W_{\mu\nu}^{gluon} \otimes \delta g$$

Initial quark distribution

Initial gluon distribution

DIS off the quark,

DIS off the gluon

are calculated with methods of Pert QCD

The Standard Approach includes the Altarelli-Parisi alias DGLAP alias Q^2- Evolution Equations and the Standard Fits for initial paron densities

Evolution Equations: Altarelli-Parisi, Gribov-Lipatov, Dokshitzer

In particular, g_1:

$$g_1(x,Q^2) = C_q(x/y) \otimes \Delta q(y,Q^2) + C_g(x/y) \otimes \Delta g(y,Q^2)$$

Evolved quark distribution

Evolved gluon distribution

Coefficient function

Coefficient function

DGLAP evolution equations

$$\frac{d\Delta q}{d\ln Q^2} = \frac{\alpha_s}{2\pi} P_{qq} \otimes \Delta q + \frac{\alpha_s}{2\pi} P_{qg} \otimes \Delta g$$

$$\frac{d\Delta g}{d\ln Q^2} = \frac{\alpha_s}{2\pi} P_{gq} \otimes \Delta q + \frac{\alpha_s}{2\pi} P_{gg} \otimes \Delta g$$

$P_{qq},\ P_{qg},\ P_{gq}, P_{gg}$ are splitting functions

Mellin transform of the splitting functions = anomalous dimensions

The Standard Approach includes the DGLAP Evolution Equations and the Standard Fits for initial parton densities. One can say that SA combines Science and Art

SCIENCE

LO splitting functions	Ahmed-Ross, Altarelli-Parisi, Sasaki,
NLO splitting functions	Floratos, Ross, Sachradja, Gonzale- Arroyo, Lopes, Yandurain, Kounnas, Lacaze, Curci, Furmanski, Petronzio, Zijlstra, Mertig, van Neerven, Vogelsang
Coefficient functions $C^{(1)}_k$, $C^{(2)}_k$	Bardeen, Buras, Muta, Duke, Altarelli, Kodaira, Efremov, Anselmino, Leader, Zijlstra, van Neerven

In DGLAP, coefficient functions and anomalous dimensions are known with LO and NLO accuracy, often at integer ω = n

LO

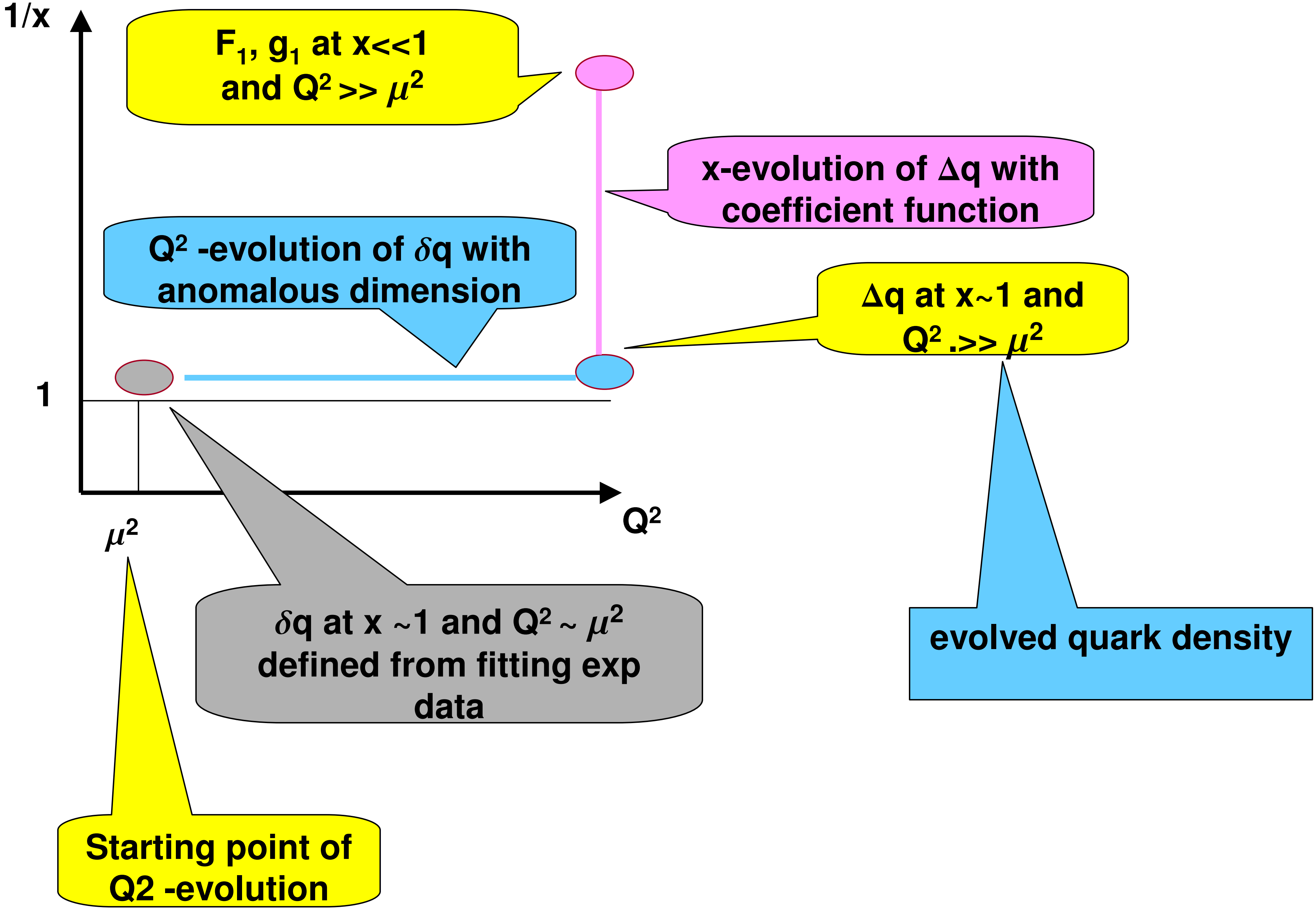
NLO

$$C(\omega) = 1 + (\alpha_s(Q^2)/2\pi)\, C^{(1)}(\omega) + \ldots$$

$$\gamma(\omega) = (\alpha_s(Q^2)/4\pi)\, \gamma^{(0)}(\omega) + (\alpha_s(Q^2)/2\pi)^2\, \gamma^{(1)}(\omega) + \ldots$$

LO

NLO


1/x
F_1, g_1 at x<<1
and $Q^2 >> \mu^2$
x-evolution of Δq with
coefficient function
Q^2 -evolution of δq with
anomalous dimension
Δq at x~1 and
Q^2 .>> μ^2
1
Q^2
μ^2
δq at x ~1 and $Q^2 \sim \mu^2$
defined from fitting exp
data
evolved quark density
Starting point of
Q2 -evolution

ART

= the art of composing the fits for initial parton densities

Altarelli-Ball-Forte-Ridolfi, Blumlein- Botcher, Leader- Sidorov- Stamenov, Hirai et al

There are different fits for initial parton densities. For example,

$$\delta q = N x^{-\alpha} [(1 - x)^{\beta} (1 + \gamma x^{\delta})]$$

$$\delta q = N [\ln^{\alpha} (1/x) + \gamma x \ln^{\beta} (1/x)]$$

Altarelli-Ball-Forte-Ridolfi,

Parameters $N, \alpha, \beta, \gamma, \delta$ should be fixed from experiment

This combination of Science and Art works well at large and small x, though strictly speaking, DGLAP is not supposed to work at the small- x region:

For example, for the simplest case of the non-singlet g_1

$$g_1^{NS}(x,Q^2) = (e_q^2/2) \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \left(\frac{1}{x}\right)^{\omega} C(\omega)\delta q(\omega) \exp\left[\int_{\mu^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \gamma(\omega, \alpha_s(k_\perp^2))\right]$$

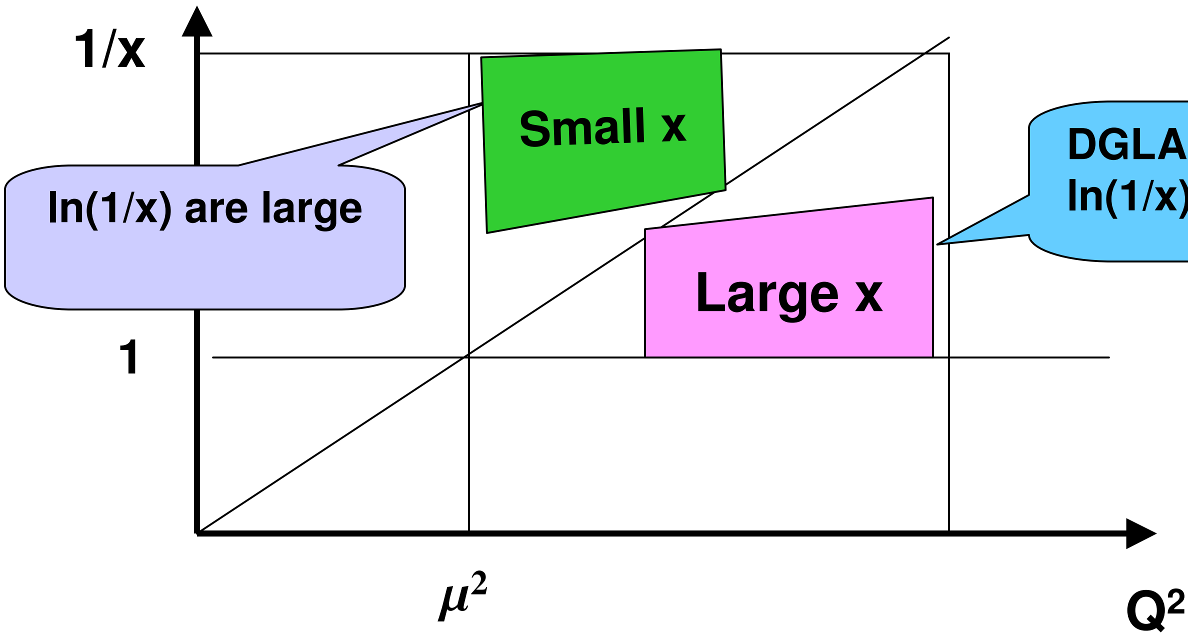
Initial quark density

Coefficient function

Anomalous dimension

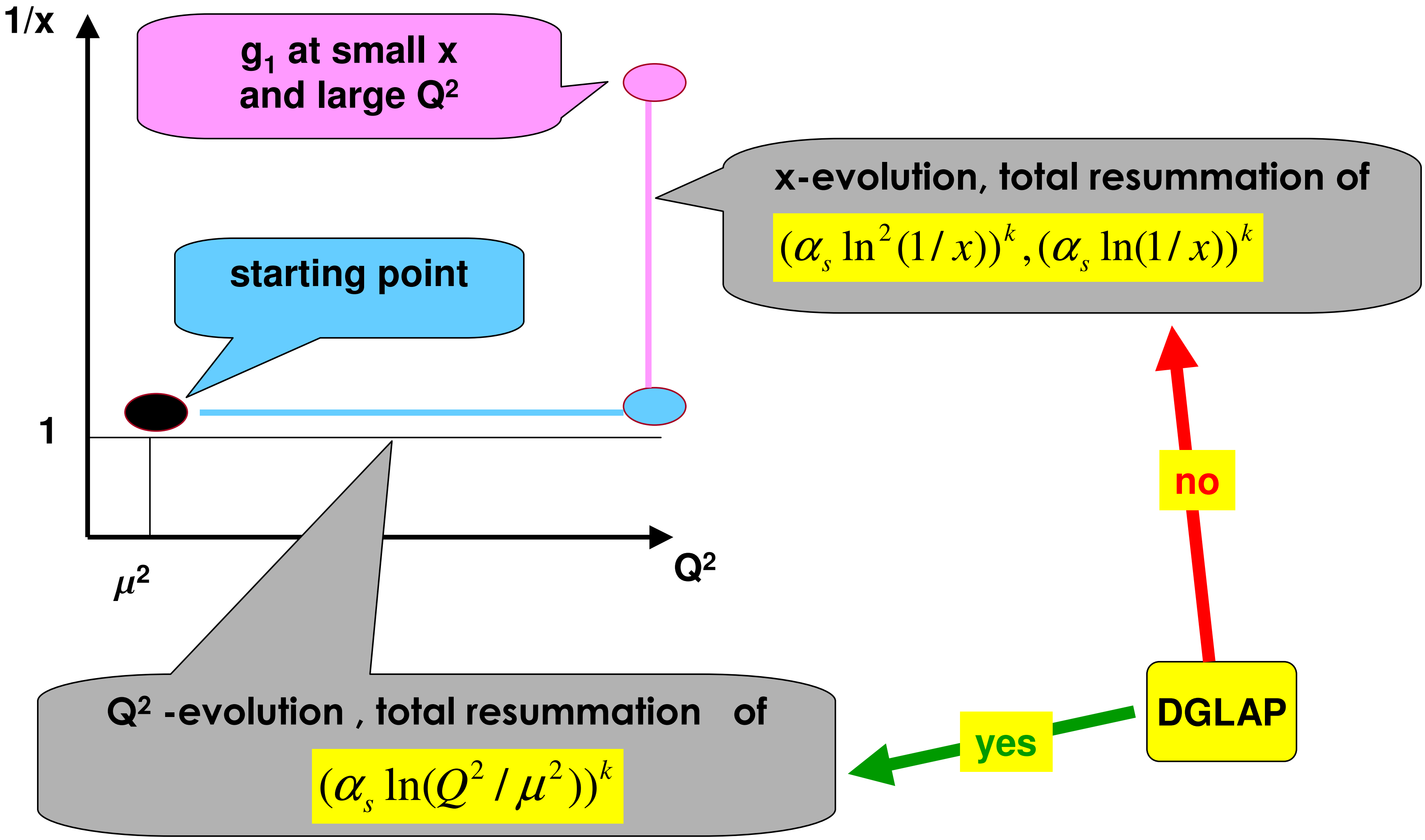
Non-Pert QCD

Pert QCD

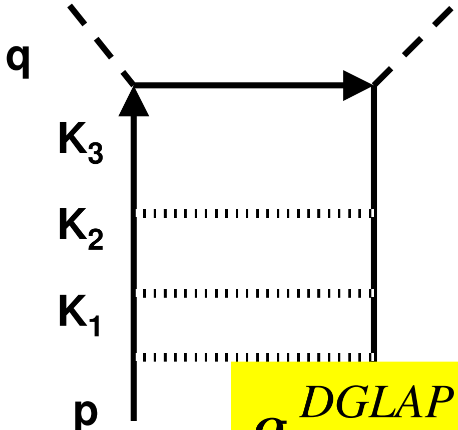


DGLAP accounts for ln(Q²) to all orders in α_s and neglects $(\alpha_s \ln^2(1/x))^k, (\alpha_s \ln(1/x))^k$ with k>2

However, these contributions become leading at small **x** and should be accounted for to all orders in the QCD coupling.



DGLAP cannot do total resummation of logs of **x** because of the DGLAP-ordering – KEYSTONE of DGLAP



DGLAP –ordering:

$$\mu^2 < k_{1\perp}^2 < k_{2\perp}^2 < k_{3\perp}^2 < Q^2$$

good approximation for large **x** when logs of **x** can be neglected. At **x << 1** the ordering has to be lifted

$$g_1^{DGLAP} \sim \exp\left[\ln(1/x) \ln\ln\left(Q^2/\Lambda_{QCD}^2\right)\right]^{1/2}$$

$$F_1^{DGLAP} \sim \left(\frac{1}{x}\right) \exp\left[\ln(1/x) \ln\ln\left(Q^2/\Lambda_{QCD}^2\right)\right]^{1/2}$$

if the initial parton densities are not singular functions of **x**

When the DGLAP –ordering is lifted and leading logarithms of x are taken into account, the asymptotics is different

The leading contributions for g_1 at small **x** are double-logarithmic (DL). Sub-leading contributions are single-logarithmic (SL)

DL contributions

SL contributions

$$(\alpha_s \ln^2(1/x))^k,$$

$$(\alpha_s \ln(1/x)\ln(Q^2/\mu^2))^k \qquad (\alpha_s \ln(1/x))^k,$$

$$k = 1,2..\infty$$

Total resummation of DL contributions =
Double-Logarithmic Approximation (DLA)

In DLA, asymptotics of g_1 is

$$g_1^{DL} \sim (1/x)^{\Delta} (Q^2/\mu^2)^{\Delta/2}$$

intercept

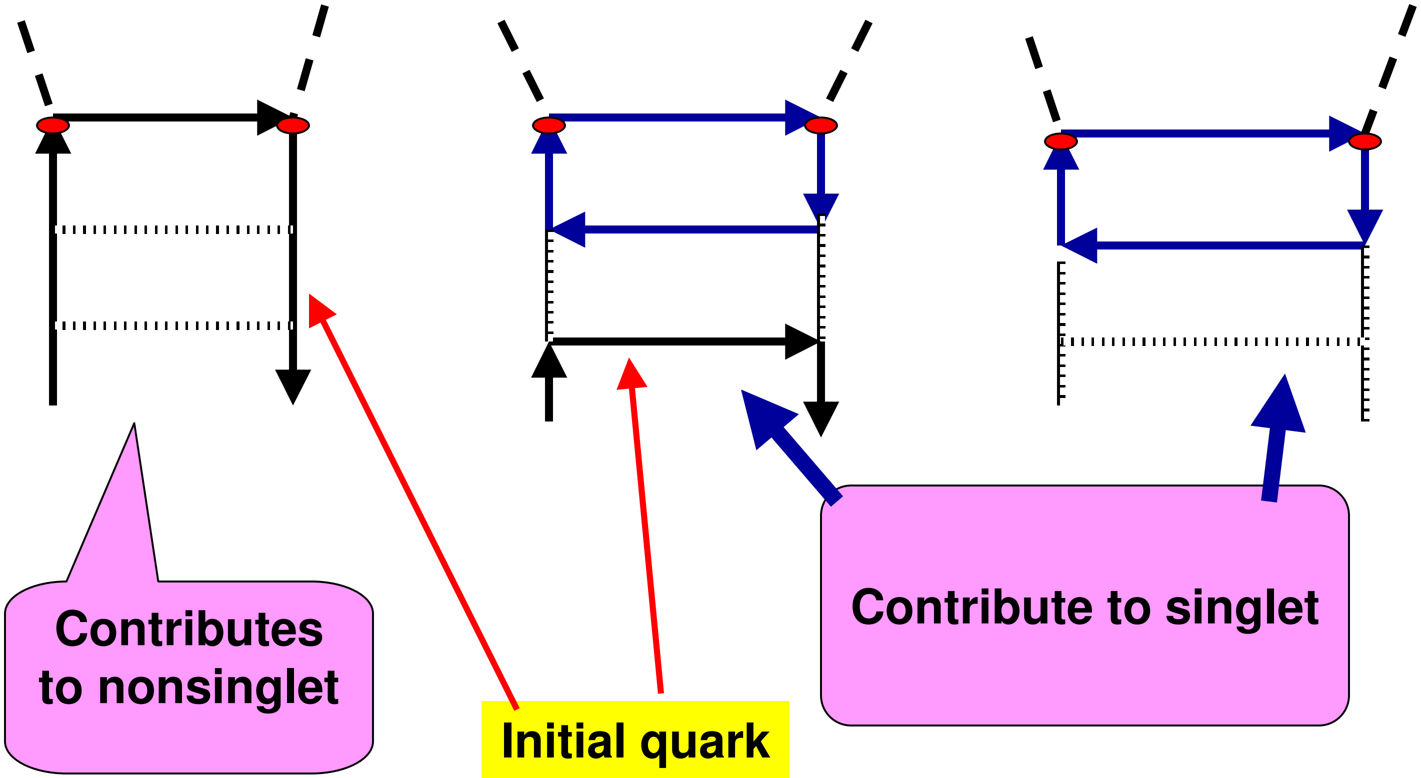
Bartels- Ermolaev-Manaenkov-Ryskin

whereas DGLAP predicts

$$g_1^{DGLAP} \sim \exp\left[\ln(1/x) \ln \ln (Q^2/\Lambda^2_{QCD})\right]^{1/2}$$

Obviously $g_1^{DL} >> g_1^{DGLAP}$ when $x \to 0$

Piece of terminology



Each structure function has both the non-singlet and singlet components:

$g_1 = g_1^{NS} + g_1^{S}$

Intercepts of g_1 in Double-Logarithmic Approximation:

non- singlet intercept

$$\Delta_{NS} = (8\alpha_s/3\pi)^{1/2},$$

singlet intercept

$$\Delta_S = 3.45\,(3\alpha_s/2\pi)^{1/2}$$

The weakest point of this approach: the QCD coupling α_s is fixed at an unknown scale.

On the contrary, DGLAP equations have always operated with running α_s

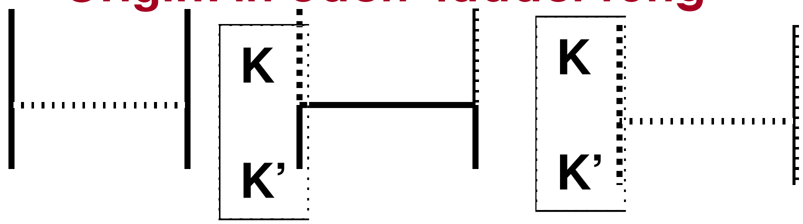
$$\alpha_s = \alpha_s(Q^2)$$

DGLAP-parameterization

Arguments in favor of the Q^2- parameterization:

Amati-Bassetto-Ciafaloni-Marchesini - Veneziano; Dokshitzer-Shirkov

Origin: in each ladder rung



$$\alpha_s = \alpha_s(k_\perp^2)$$

DGLAP-parameterization

Ermolaev-Greco-Troyan

However, such a parameterization is good for large x only. At small **x** :

$$\alpha_s = \alpha_s((k-k')^2) \approx \alpha_s((k_\perp^2 + k'^2_\perp)/x) \approx \alpha_s(k_\perp^2)$$

time-like argument

Participates in the Mellin transform

When DGLAP-ordering is used and x ~1

DGLAP -parametrization

$$\alpha_s = \alpha_s(Q^2)$$

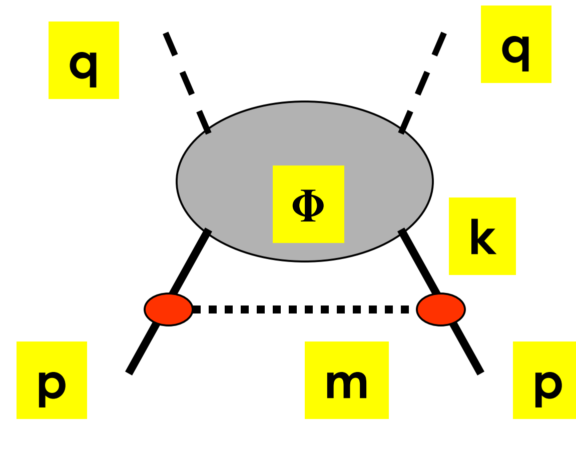
space-like argument,

no Mellin transform

DIS structure functions obey the Bethe-Salpeter equation:

$$\Phi = \Phi_{born} + $$

w=2pq



$$\int dk_\perp^2 d\beta dm^2 \Phi(w, Q^2, \beta, k_\perp^2, m^2) \frac{1}{\beta m^2 + k_\perp^2} \mathrm{Im}\left(\frac{\alpha_s(m^2)}{m^2 + i\varepsilon} \right)$$

$m^2 > 0$

$$\alpha_s(m^2) = \frac{1}{b \ln(-m^2 / \Lambda_{QCD}^2)} = \frac{1}{b\,[\ln(m^2 / \Lambda_{QCD}^2) - i\pi]}$$

$$1 > \beta > x + \frac{k_\perp^2 (1 - x)}{w - m^2}$$

Integral over m² is interpreted as a dispersion relation:

Dokshitzer-Shirkov

$$\approx \frac{1}{\pi}\int_0^\infty dm^2 \frac{1}{\beta m^2 + k_\perp^2} \operatorname{Im}\left(\frac{\alpha_s(m^2)}{m^2}\right) = \frac{1}{k_\perp^2}\alpha_s(-k_\perp^2/\beta) \approx \frac{\alpha_s(-k_\perp^2)}{k_\perp^2}$$

$$= \frac{1}{k_\perp^2}\frac{1}{b\ln(k_\perp^2/\Lambda_{QCD}^2)}$$

fraction of longitudinal moment

when x close to 1

However this interpretation is valid for large x only, when

$$\beta_{\min} = x + \frac{k_\perp^2(1-x)}{w-m^2} \approx x$$

and cannot be used for small x

At x<<1 and with the leading logarithmic accuracy one can Integrate over m²:

$$\int_0^\infty dm^2 \frac{1}{\beta m^2 + k_\perp^2} \mathrm{Im}\left(\frac{\alpha_s(m^2)}{m^2}\right) \approx \frac{1}{k_\perp^2} \int_0^{k_\perp^2/\beta} dm^2 \, \mathrm{Im}\left(\frac{\alpha_s(m^2)}{m^2}\right) =$$

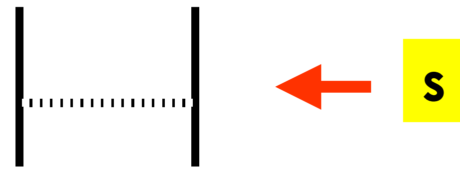
$$\frac{1}{k_\perp^2} \left[\frac{1}{b} \arctan\left(\frac{1}{\pi} \ln\left(\frac{k_\perp^2}{\beta \Lambda_{QCD}^2}\right)\right)\right]$$

plays the role of α_s when $x << 1$

However, it is better to do the Mellin transform

Quark-quark scattering amplitude in the Born approximation

$$M_B(s) = \alpha_s(s)\frac{s}{s-\mu^2+i\varepsilon}$$



IR cut-off

The minus sign for respecting the analyticity

$$\alpha_s(s) = \frac{1}{b\ln(-s/\Lambda^2_{QCD})} =$$

$$\frac{1}{b[\ln(s/\Lambda^2_{QCD}) - i\pi]} = \frac{\ln(s/\Lambda^2_{QCD}) + i\pi}{b[\ln^2(s/\Lambda^2_{QCD}) + \pi^2]}$$

The coupling participates in the Mellin transform

$$M_B = \alpha_s(s) \frac{s}{s - \mu^2 + i\varepsilon} \to \frac{A(\omega)}{\omega}$$

where

$$\alpha_s(s) \to A(\omega) = \frac{1}{b}\left[\frac{\eta}{\eta^2 + \pi^2} - \int_0^\infty d\rho \frac{\exp(-\omega\rho)}{(\rho + \eta)^2 + \pi^2}\right]$$

with

$$\eta = \ln(\mu^2 / \Lambda_{QCD}^2)$$

It is valid when

$$\mu^2 > \Lambda_{QCD}^2$$

This restriction guarantees the applicability of Pert QCD

Expression for the non-singlet g_1 at large Q^2: $Q^2 >> 1$ GeV^2

Initial quark density

$$g_1^{NS} = \frac{e_q^2}{2} \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} \left(\frac{1}{x}\right)^{\omega} \left(\frac{\omega}{\omega - H(\omega)}\right) \delta q(\omega) \left(\frac{Q^2}{\mu^2}\right)^{H(\omega)}$$

Coefficient function

Anomalous dimension

New coefficient function and anomalous dimension sum up leading logarithms to all orders in α_s

Compare our non-singlet anomalous dimension to the LO DGLAP one:

expand C and H into series in $1/\omega$

$$H = \frac{A(\omega)C_F}{2\pi}\left[\frac{1}{\omega} + \frac{1}{2}\right] + \ldots$$

coincide, save the treatment of α_s

$$\gamma_{NS}^{\text{LO DGLAP}} = \frac{\alpha_s(Q^2)C_F}{2\pi}\left[\frac{1}{n(n+1)} + \frac{3}{2} - S_2(n)\right] \approx \frac{\alpha_s(Q^2)C_F}{2\pi}\left[\frac{1}{n} + \frac{1}{2} + O(n)\right]$$

when n < 1

where

$$S_k(n) = \sum_{j=1}^{n} \frac{1}{j^k}$$

small/large x ⟷ **small/large n**

Compare our coefficient function and the NLO DGLAP one

$$C = \frac{\omega}{\omega - H(\omega)} = 1 + \frac{A(\omega) C_F}{2\pi}\left[\frac{1}{\omega^2} + \frac{1}{2\omega}\right] + \ldots$$

coincide, save the treatment of α_s

LO

NLO

$$C_{NS}^{\mathrm{DGLAP}} = 1 + \frac{\alpha_s(Q^2) C_F}{2\pi}\left[\frac{1}{n^2} + \frac{1}{2n} + \frac{1}{2n+1} - \frac{9}{2} + \left(\frac{3}{2} - \frac{1}{n(1+n)}\right) S_1(n) + S_1^2(n) - S_2(n)\right]$$

when n < 1

$$\approx \frac{\alpha_s(Q^2) C_F}{2\pi}\left[\frac{1}{n^2} + \frac{1}{2n} + O(n)\right]$$

Expression for the singlet g_1 at large Q^2:

$$g_1^S = \frac{< e_q^2 >}{2} \int \frac{d\omega}{2\pi i} \left(\frac{1}{x} \right)^{\omega}$$

$$\left[\left(C_q^{(+)} \delta q + C_q^{(+)} \delta g \right) \left(\frac{Q^2}{\mu^2} \right)^{\Omega^{(+)}} + \left(C_q^{(-)} \delta q + C_q^{(-)} \delta g \right) \left(\frac{Q^2}{\mu^2} \right)^{\Omega^{(-)}} \right]$$

$$\Omega^{(+)} > \Omega^{(-)}$$

Large Q^2 means

$$Q^2 > \mu^2; \;\; \mu \approx 5 \text{ GeV}$$

Small –x symptotics of g_1: when $x \to 0$, the saddle-point method leads to

$$g_1^{NS} \sim \frac{e_q^2}{2} (1/x)^{\Delta_{NS}} \left(Q^2/\mu^2\right)^{\Delta_{NS}/2} \delta q$$

Nonsinglet intercept $\Delta_{NS} = 0.42$

At large x, g_1^{NS} and g_1^{S} are positive

$\delta q > 0$ → $g_1^{NS} > 0$ **In the whole range of x at any Q^2**

Asymptotics of the singlet g_1 are more involved

$$g_1^S \sim \frac{\langle e_q^2 \rangle}{2} S(\Delta_S)\,(1/x)^{\Delta_S}\left(Q^2/\mu^2\right)^{\Delta_S/2}$$

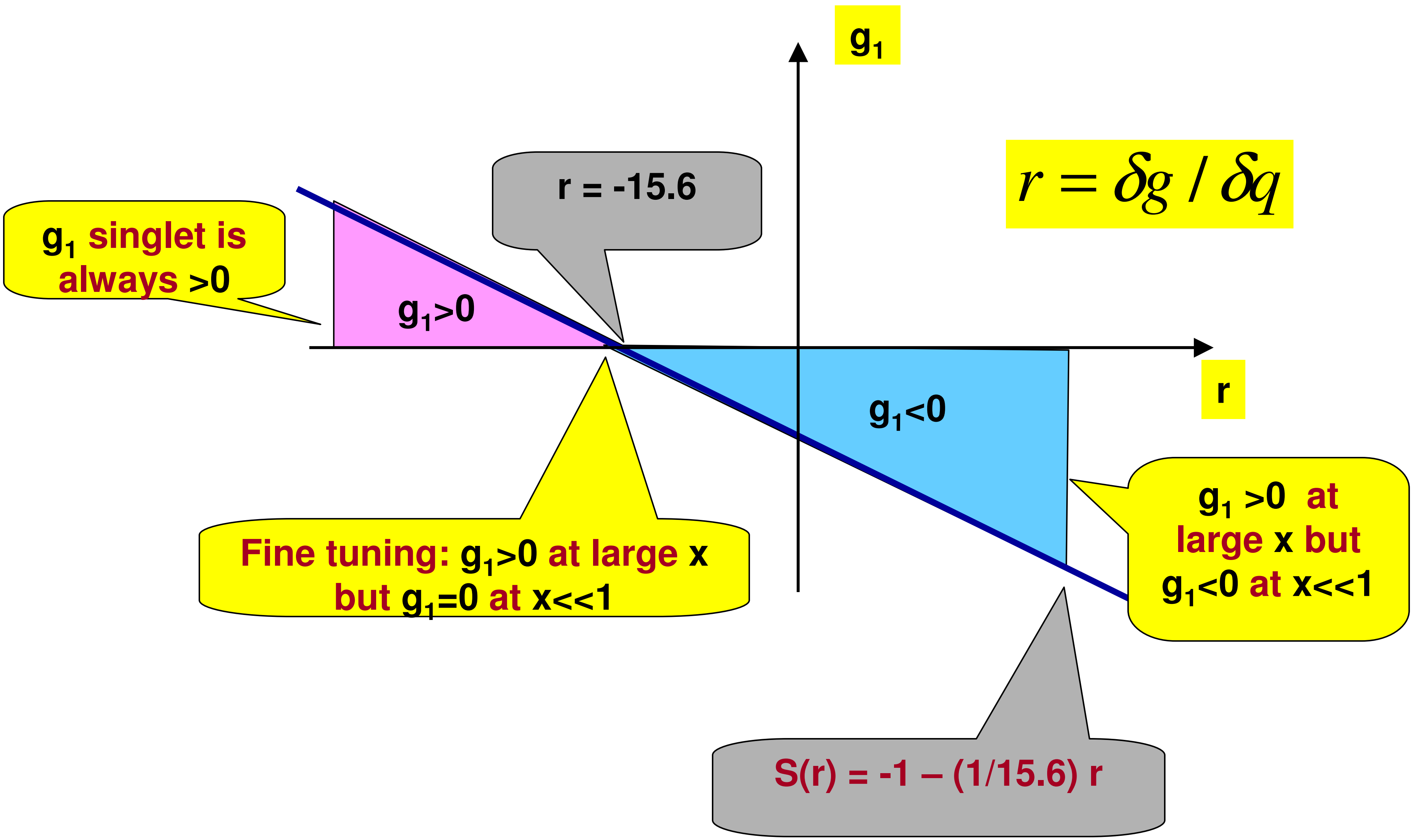
With intercept $$\Delta_S = 0.86$$

and $$S(\Delta_S) = -\delta q - 0.064\ \delta g$$

Interplay between the quark and gluon densities can lead to different sign of g_1 singlet at $x<<1$

Warning: asymptotic expressions $g_1 \sim (1/x)^{\Delta}$ are reliable at x<10-5

At large x, g_1 singlet is positive . When x--> 0, the sign of asymtpotics of the singlet g_1 depends on the ratio between the initial parton densities



Values of the intercepts perfectly agree with results of several groups who fitted experimental data.

non-singlet intercept	Soffer-Teryaev, Kataev-Sidorov-Parente, Kotikov-Lipatov-Parente-Peshekhonov-Krivokhijine-Zotov,
singlet intercept	Kochelev-Lipka-Vento-Novak-Vinnikov

Anatomy of the singlet intercept

A. Graphs with gluons only: $\Delta_S = 1.1$ → violates unitarity → similar to LO BFKL

B. All graphs $\Delta_S = 0.86$ → No violation of unitarity

Comparison of our results to DGLAP at finite x –no asymptotic formulae used

Comparison depends on the assumed shape of initial parton densities.

The simplest option: use the bare quark input

$$\delta q(x) = \delta(\mathrm{x}) \quad \Longleftrightarrow \quad \delta q(\omega) = 1$$

in x- space | in Mellin space

Numerical comparison shows that the impact of the total resummation of logs of x becomes quite sizable at x = 0.05 approx.

Hence, DGLAP should have Failed at x < 0.05.
However, it does not take place.

In order to understand what could be the reason for success of DGLAP at small x, let us consider in more detail

standard fits for initial parton densities.

$$\delta q(x) = N\, x^{-\alpha}[(1+\gamma x^{\delta})(1-x)^{\beta}]$$

Altarelli-Ball-Forte-Ridolfi

normalization

singular factor

regular factors

parameters $\alpha \approx 0.58,\ \beta \approx 2.7, \gamma \approx 34.3,\ \delta \approx 0.75$

are fixed from fitting experimental data at large x

In the Mellin space this fit is

$$\delta q(\omega) = N[(\omega-\alpha)^{-1} + \sum_{k=1}^{\infty} c_k ((\omega+k-\alpha)^{-1} + \gamma(\omega+k+1-\alpha)^{-1})]$$

Leading pole
α=0.58 >0

Non-leading poles
-k +α<0

the small-x DGLAP asymptotics of g_1 is (inessential factors dropped)

$$g_1^{DGLAP} \sim (1/x)^{\alpha}$$

phenomenology

Comparison it to our asymptotics

$$g_1 \sim (1/x)^{\Delta_{NS}}$$

calculations

shows that the singular factor in the DGLAP fit mimics the total resummation of ln(1/x) . However, the value α = 0.58 sizably differs from our non-singlet intercept =0.4

Although our and DGLAP asymptotics lead to the x- behavior of Regge type, they predict different intercepts for the x- dependence and different Q^2 -dependence:

our calculations

$$g_1 \sim (1/x)^{\Delta} \left(Q^2/\mu^2\right)^{\Delta/2}$$

whereas DGLAP predicts the steeper x-behavior and the flatter Q^2 -behavior:

DGLAP

$$g_1^{DGLAP} \sim (1/x)^{\alpha} (\ln Q^2)^{\gamma(\alpha)}$$

x-asymptotics was checked with extrapolating available exp data to $x \to 0$. It agrees with our values of Δ Contradicts DGLAP

our and the DGLAP Q^2 –asymptotics have not been checked yet.

Common opinion: the total resummation is not relevant at available x

Actually: the resummation has always been accounted for through the standard fits, however without realizing it

Common opinion: fits for δq are singular but defined and large x,
then convoluting them with coefficient functions weakens the singularity

$$C(x,y)\otimes\delta q(y)=\Delta q(x)$$

initial — x-evolved

Obviously, it is not true:
They both are singular equally

Structure of DGLAP fit once again:

$$\delta q(x)=N\,\mathrm{x}^{-\alpha}[(1+\gamma\mathrm{x}^{\delta})(1-x)^{\beta}]$$

Can be dropped when ln(x) are resummed

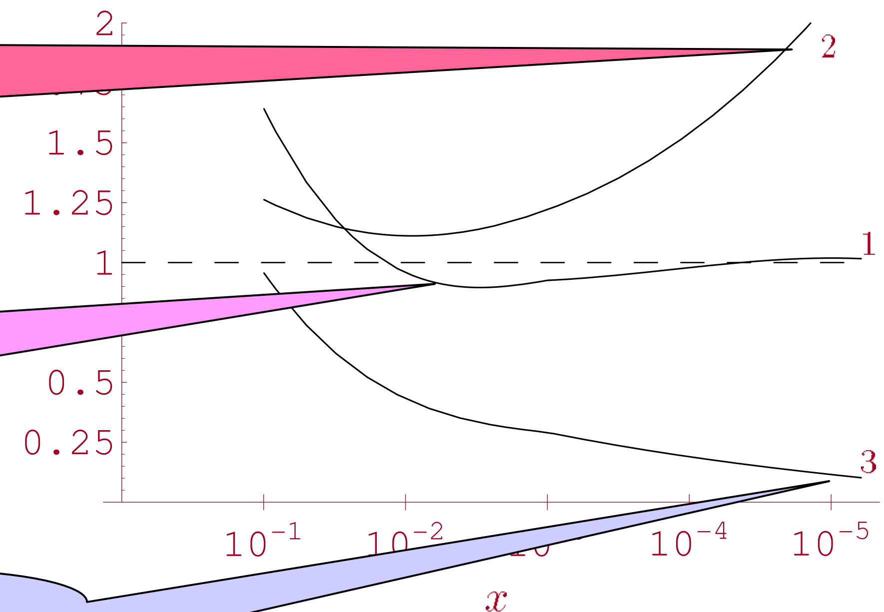
x-dependence is weak at x<<1 and can be dropped

Therefore at x << 1

$$\delta q(x)\approx \mathrm{N}(1+\mathrm{ax})$$

Numerical comparison of DGLAP with our approach at small but finite x, using the same DGLAP fit for initial quark density.

$R = g_1^{our}/g_1^{DGLAP}$



x

Only regular factors in g_1^{our} and g_1^{DGLAP}

Regular term in g_1^{our} vs regular + singular in g_1^{DGLAP}

Whole fit in g_1^{our} and g_1^{DGLAP}: regular + singular

Comparison between DGLAP and our approach at small x

DGLAP

coeff functions and anom dimensions are calculated with two-loop accuracy

To ensure the Regge behavior, singular in x terms in initial partonic densities are used

Equivalent to inserting a phenomenological asymptotic factor into expressions for g_1 but

warning: using asymptotic formulae for g_1 is unreliable at $x > 10^{-5}$

our approach

coeff functions and anom dimensions sum up DL and SL terms to all orders

Regge behavior is achieved automatically, even when the initial densities are regular in x

Asymptotics of g_1 are never used in expressions for g_1 at finite x

Comparison between DGLAP and our approach at any x

DGLAP	our approach
Good at large x because includes exact two-loop calculations but bad at small x as lacks the total resummaion of ln(x)	Good at small x , includes the total resummaion of ln(x) but bad at large x because neglects some contributions essential in this region

WAY OUT – synthesis of our approach and DGLAP

1. Expand our formulae for coefficient functions and anomalous dimensions into series in the QCD coupling
2. Replace the first- and second- loop terms of the expansion by corresponding DGLAP –expressions

Our expressions

$$H(\omega) = (1/2)[\omega - (\omega^2 - B(\omega))]^{1/2} \quad C(\omega) = \omega/(\omega - H(\omega))$$

anomalous dimension

coefficient function

First tems of their expansions into the perturbation series

$$H_1 = \frac{A(\omega)C_F}{2\pi}\left[\frac{1}{\omega} + \frac{1}{2}\right] \quad C_1 = \frac{A(\omega)C_F}{2\pi}\left[\frac{1}{\omega^2} + \frac{1}{2\omega}\right]$$

New, "synthetic" formulae:

$$h = H - H_1 + H_{LO\ DGLAP} \quad c = C - C_1 + C_{LO\ DGLAP}$$

New, "synthetic" formulae accumulate all advantages of the both approaches and should equally be good at large and small x.

New fits should not involve singular factors

Taken from wwwcompass.cern.ch

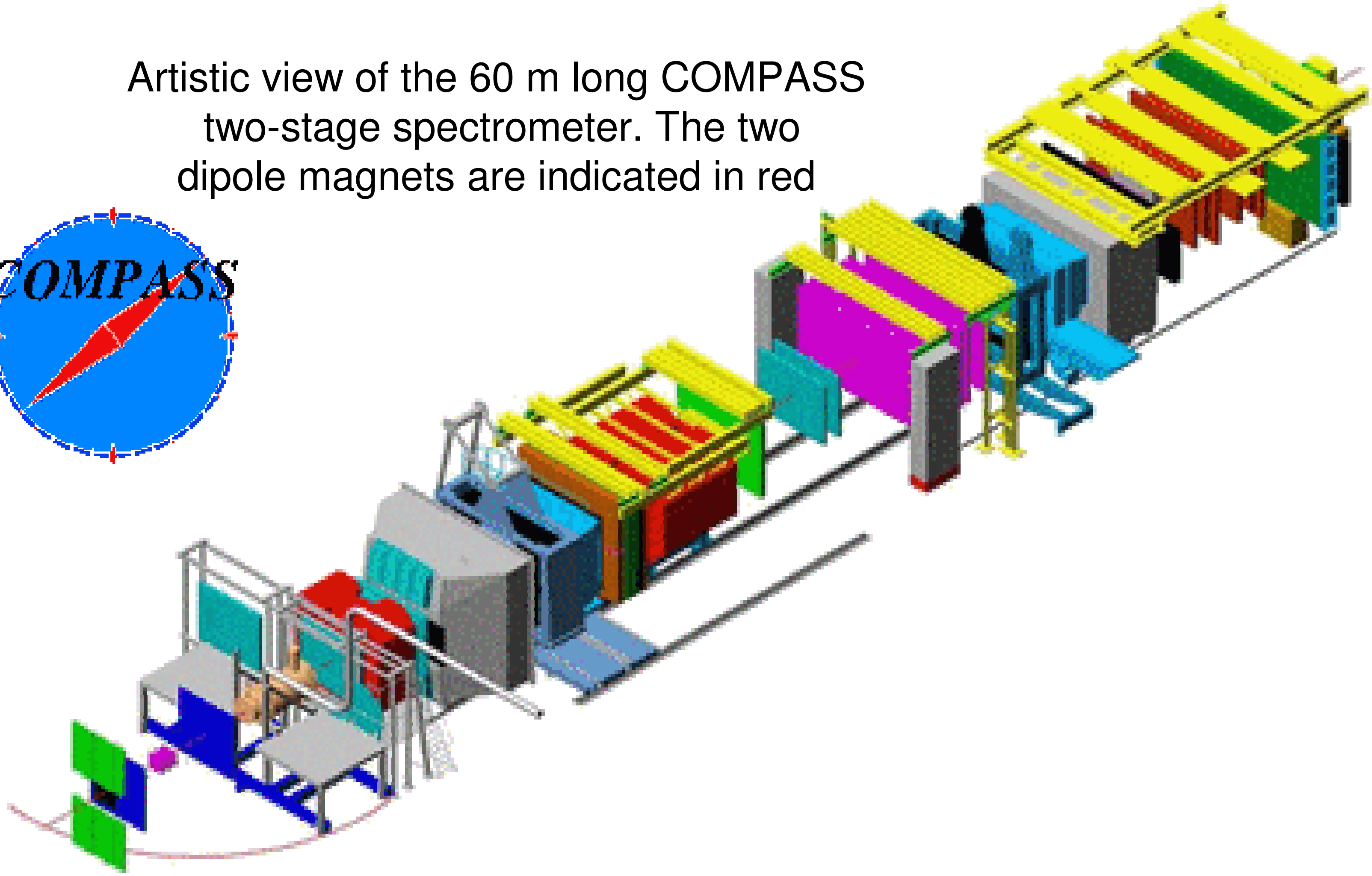
COMPASS is a high-energy physics experiment at the Super Proton Synchrotron (SPS) at CERN in Geneva, Switzerland. The purpose of this experiment is the study of hadron structure and hadron spectroscopy with high intensity muon and hadron beams.
On February 1997 the experiment was approved conditionally by CERN and the final Memorandum of Understanding was signed in September 1998. The spectrometer was installed in 1999 - 2000 and was commissioned during a technical run in 2001. Data taking started in summer 2002 and continued until fall 2004. After one year shutdown in 2005, COMPASS will resume data taking in 2006.
Nearly 240 physicists from 11 countries and 28 institutions work in COMPASS

Taken from wwwcompass.cern.ch

COMPASS

COmmon Muon Proton Apparatus for Structure and Spectroscopy

Artistic view of the 60 m long COMPASS two-stage spectrometer. The two dipole magnets are indicated in red



COMPASS operates with small Q^2 ($Q^2 < 10^{-1}$ GeV^2) and small $x \sim 10^{-3}$

In order to generalize our results to the region of small Q^2, one should remember that $\ln(Q^2/\mu^2)$ is the result of the integration $\int_{\mu^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}$

obained for large Q^2 with logarithmic accuracy.
For arbitrary Q^2 one can use the prescription:

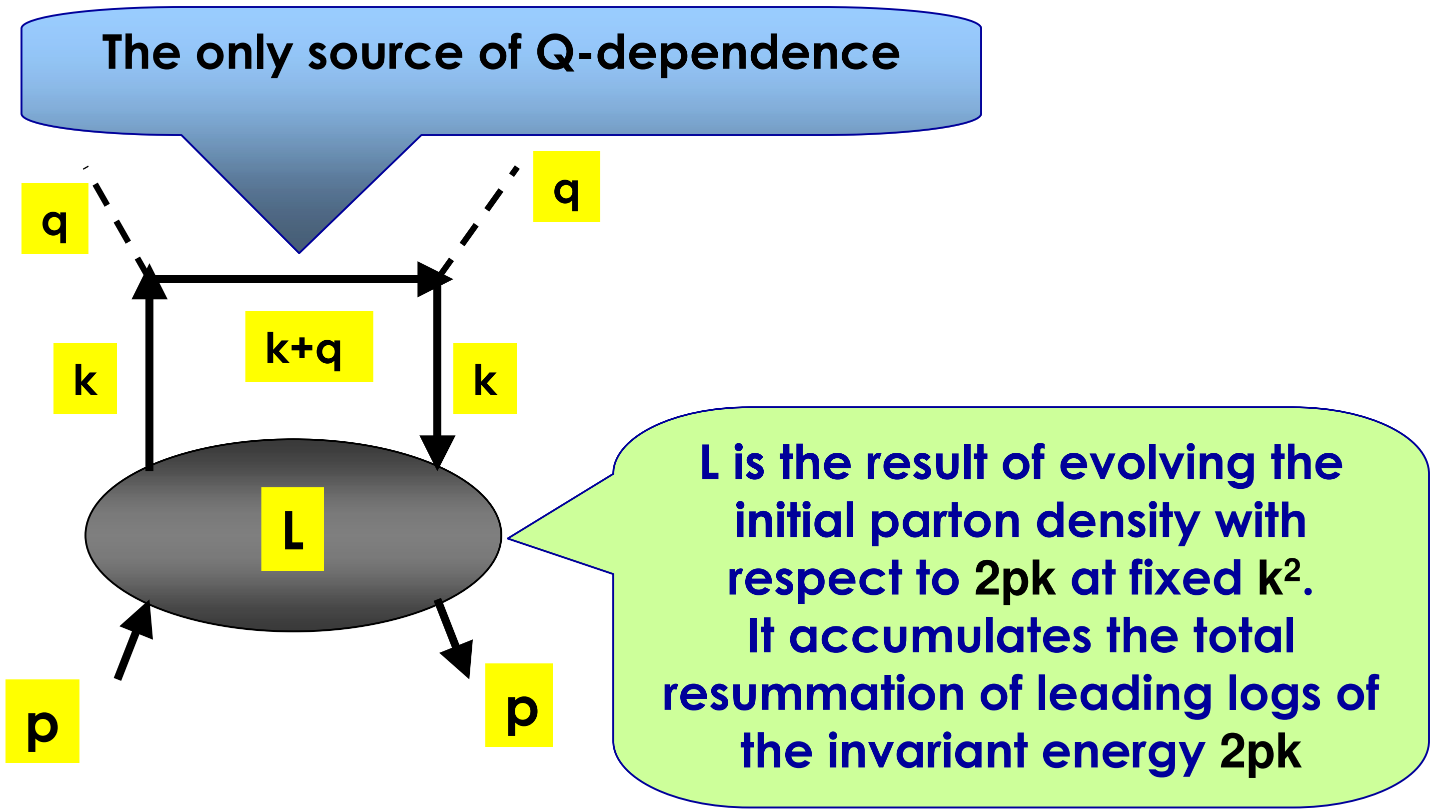
$$\mathrm{Q}^2 \to \mathrm{Q}^2 + \mu^2 \quad \Rightarrow \quad \mathrm{x} \to \bar{x} = (\mathrm{Q}^2 + \mu^2)/2\mathrm{pq} = x + z$$

Infrared cut-off

Similar to the Nachtmann variable

Obviously, g_1 obeys the Bete-Salpeter equation:

$g_1=$



$$g_1 = \int \frac{d^4k\, k_\perp^2}{(k^2+\mu^2)^2}\delta(k^2+2qk-(Q^2+\mu^2))L(2pk,k^2,\mu^2)$$

Using the Sudakov parameterization

$$k = \alpha q + (\beta + x\alpha) p + k_\perp$$
$$\approx \alpha q + \beta p + k_\perp$$

leads o the following integral representation for g_1 at $x \ll 1$

$$g_1 = \int_0^w \frac{dk_\perp^2}{k_\perp^2 + \mu^2} L(Q^2 + \mu^2, w, k_\perp^2, \mu^2)$$

This integral can approximately be calculated at $Q^2 \gg \mu^2$ and at $Q^2 \ll \mu^2$. The both cases can approximately be written through the shift $Q^2 \Rightarrow Q^2 + \mu^2$

It leads to new expressions: non-singlet g_1 at small Q^2

$$z = \frac{\mu^2}{2pq} >> x = \frac{Q^2}{2pq}$$

$$g_1^{NS} = \frac{e_q^2}{2}\int \frac{d\omega}{2\pi i}\left(\frac{1}{z+x}\right)^{\omega}\left(\frac{\omega}{\omega - H(\omega)}\right)\delta q(\omega)\left(\frac{\mu^2+Q^2}{\mu^2}\right)^{H(\omega)}$$

weak x -dependence

Anomalous dimension

weak Q^2 -dependence

Coefficient function

Initial quark density

Singlet g_1 at small Q^2

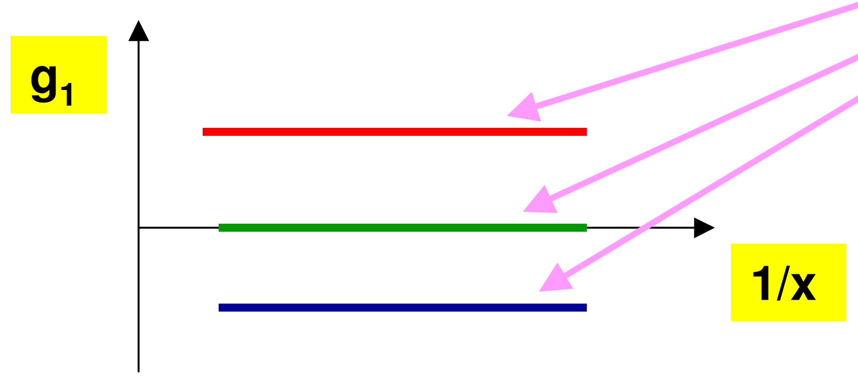
$$z = \frac{\mu^2}{2pq}, \quad x = \frac{Q^2}{2pq}$$

$$g_1^S = \frac{<e_q^2>}{2}\int \frac{d\omega}{2\pi i}\left(\frac{1}{z+x}\right)^{\omega}\left[C_q \delta q + C_g \delta g\right]$$

$$C_g = C_g^{(+)}\left(\frac{\mu^2+Q^2}{\mu^2}\right)^{\Omega^{(+)}} + C_g^{(-)}\left(\frac{\mu^2+Q^2}{\mu^2}\right)^{\Omega^{(-)}}$$

$$C_q = C_q^{(+)}\left(\frac{\mu^2+Q^2}{\mu^2}\right)^{\Omega^{(+)}} + C_q^{(-)}\left(\frac{\mu^2+Q^2}{\mu^2}\right)^{\Omega^{(-)}}$$

when $Q^2 << \mu^2$ both x- and Q^2- dependences are flat, even for x<<1.



Location of the line is determined by the z-dependence

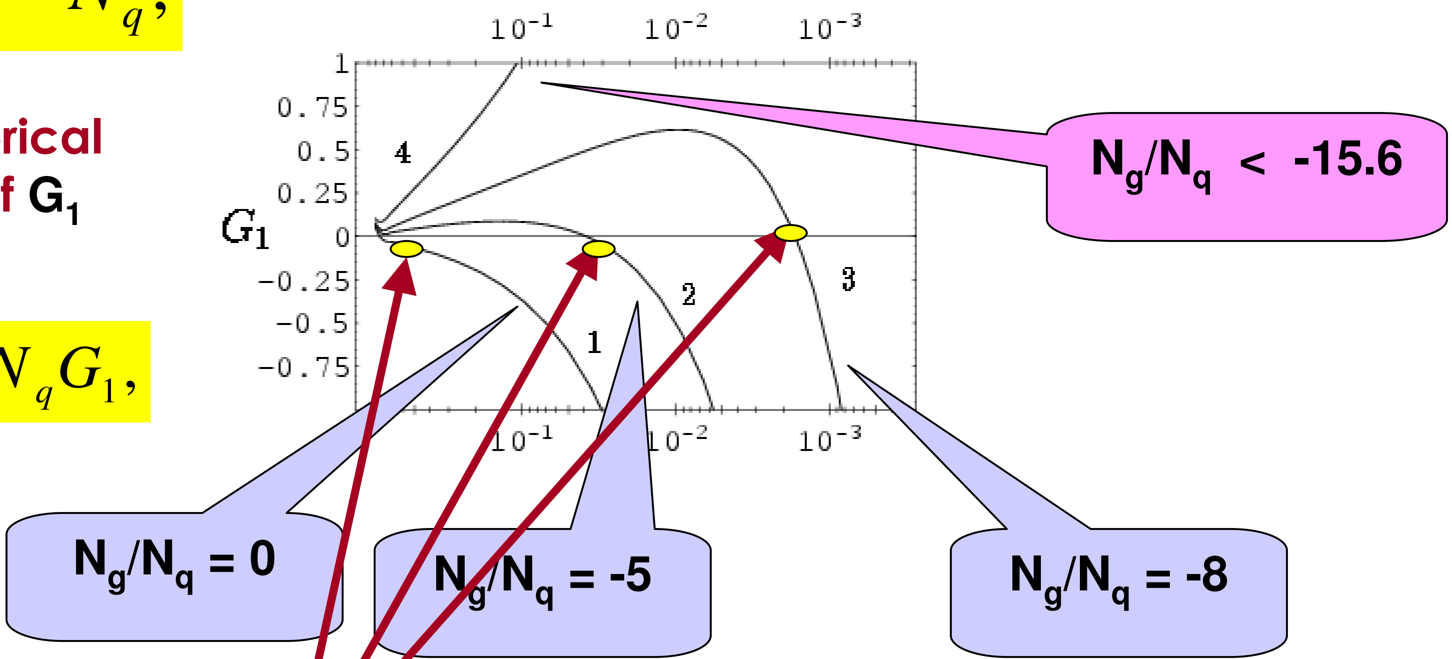
$$g_1(z) = \left(\frac{e_q^2}{2}\right)\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\left(\frac{1}{z}\right)^{\omega}\left[C_q(\omega)\delta q + C_g(\omega)\delta g\right]$$

Approximating

$$\delta q \approx N_q, \delta q \approx N_q,$$

perform numerical calculations of G_1

$$g_1 = (e_q^2/2)N_q G_1,$$



Position of the turning point is sensitive to N_g/N_q , so the experimental detection of it will allow to estimate Ng/Nq

Power Corrections to non-singlet g_1

Leading twist contribution

mass scale: $Q^2 > M^2$

$$g_1(x, Q^2) = g_1^{LT}(x, Q^2)\left[1 + \sum_k C_k \left(\frac{M^2}{Q^2}\right)^k\right]$$

Power corrections

PC are supposed to come from higher twists.
No satisfactory theory
is known for the higher twists

Standard way of obtaining PC from experimental
data at small x: Leader-Stamenov- Sidorov

Compare experimental data to predictions of the Standard Approach
and assign the discrepancy to the impact of PC

$$g_1^{LT} = g_1^{DGLAP}$$

Counter-arguments:

1. **DGLAP, the main ingredient of SA, is unreliable at small x, so comparing experiment to it is not productive**
2. **SA cannot explain why PC appear at $Q^2 > 1$ GeV^2 only and predict what happens at smaller Q^2**

Our approach :

$$g_1^{NS} = \frac{e_q^2}{2} \int \frac{d\omega}{2\pi i} \left(\frac{\mathrm{w}}{\mu^2 + Q^2} \right)^{\omega} C(\omega)\, \delta q(\omega) \left(\frac{\mu^2 + Q^2}{\mu^2} \right)^{H(\omega)}$$

where w = 2pq and Q^2 can be large or small, μ = 1 GeV

As μ **=1 GeV, at $Q^2 > 1$ GeV^2 expansion into series is**

$$g_1^{NS} =$$

$$\frac{e_q^2}{2}\int \frac{d\omega}{2\pi i}\left(\frac{w}{Q^2}\right)^{\omega} C(\omega)\,\delta q(\omega)\left(\frac{\mu^2}{Q^2}\right)^{H(\omega)}$$

$$\left[1+\sum_{k=1} T_k(\omega)\left(\frac{\mu^2}{Q^2}\right)^{k}\right]$$

Power corrections

Leading contribution For g_1^{NS}

When $Q^2 < 1$ GeV^2, PC are different:

$$g_1^{NS} = \frac{e_q^2}{2} \int \frac{d\omega}{2\pi i} \left(\frac{w}{\mu^2} \right)^{\omega} C(\omega)\, \delta q(\omega)$$

$$\left[1 + \sum_{k=1} T_k(\omega) \left(\frac{Q^2}{\mu^2} \right)^k \right]$$

Power corrections

Leading contribution for g_1^{NS} does not depend on Q^2

These power corrections have perturbative origin and should be accounted in the first place. Only after that one can estimate a genuine impact of higher twist contributions

Conclusion

Infrared Evolution Equations Approach is a simple and efficient instrument for performing total resummations of Double- Log and a certain part of Single- Log contributions to processes in QCD, QED and electroweak reactions in the hard and Regge kinematics

Being applied to the Polarized DIS, this approach can be regarded as an alternative to the Standard Approach

Standard Approach

DGLAP was originally developed for operating at the region where both x and Q^2 are large. Basic ingredients of the DIS structure functions – coefficient functions and splitting functions (anomalous dimensions) are calculated in DGLAP in the first and second loops. By construction, DGLAP describes the Q^2–evolution but cannot describe the x–evolution. Accounting for the x–evolution is especially important in the small-x region.

In order to extend DGLAP to the region of small **x** and large **Q^2**, it have been complemented with rather complicated expressions for the initial parton densities δ**q** and δ**g** found from fitting experimental data.

DGLAP + Standard fits form Standard Approach (SA). SA describes DIS at large **Q^2** and arbitrary **x**.

We have obtained the model-independent description of g_1 combining total resummation of leading logarithmic contributions and DGLAP expressions. Represent g_1 at arbitrary x and Q^2 .

DGLAP agrees with experimental data only when special expressions for initial parton densities are used. They include singular factors, though DGLAP offers no theoretical explanation of the origin of the factors

Actually, the singular factors mimic total resummation of leading logarithms When the resummatiion is accounted for, the expressions for initial parton densities can be simplified down to constants

The region of small Q^2 is also beyond the reach of SA. We predict that g_1 at small Q^2 is almost independent of x, even at $x \ll 1$. Instead, it depends on $2pq$ only. At a certain relation between the initial quark and gluon densities, g_1 can be pretty close to zero in the range of $2pq$ investigated now experimentally by COMPASS.

Besides genuine PC from higher twists, there are perturbative PC. They should be accounted in the first place and only after that the impact of higher twists can be estimated